Physica A 389 (2010) 2915–2919

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

The tensor renormalization group for pure and disordered two-dimensional lattice systems

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ARTICLE INFO

Article history:
Received 4 February 2010
Available online 11 February 2010

Keywords:
Renormalization group
Disorder

ABSTRACT

The tensor renormalization group (TRG) is a powerful new approach for coarse-graining classical two-dimensional (2D) lattice Hamiltonians. It uses the intuitive framework of traditional position space renormalization group methods – analyzing flows in the space of Hamiltonian parameters – but can be systematically improved to yield thermodynamic properties at much higher precision. We present initial results demonstrating that the TRG can be generalized to quenched random systems, applying it to obtain the phase diagram of a bond-diluted triangular lattice Ising ferromagnet. This opens a variety of potential future applications, most prominently spin glasses.

1. Introduction

The recently developed tensor renormalization group (TRG) [1] for classical 2D lattice systems represents a significant step forward for position space renormalization techniques. Compared to older approximate coarse-graining approaches based on system-specific heuristics [2–7], the TRG has several advantages: (i) it applies to any lattice model that can be formulated in terms of local Hamiltonians expressed as tensors at each lattice site, including the Ising, Potts, and various vertex models; (ii) it can be systematically improved in a straightforward fashion to converge to the exact free energies and phase transition temperatures; (iii) though its numerical accuracy is competitive with techniques like finite-size scaling of large systems, it can be interpreted just like the traditional renormalization group (RG)—by analyzing flows in a multidimensional parameter space [8].

The TRG has already proven its usefulness for frustrated and unfrustrated classical Ising models on the triangular [1] and Shastry–Sutherland [9] lattices. But these are only the first steps in a much broader array of potential applications, which include the thermodynamic characterization of complicated multicritical systems. In the present work we briefly outline the original method, and show preliminary results for a promising extension: applying the TRG to quenched random systems, using bond percolation in a triangular lattice Ising ferromagnet as a test case [10].

2. TRG for pure systems

We start with a review of the TRG method for pure lattice systems (more details can be found in [1,8]). Assuming that we have a Hamiltonian with local interactions that can be expressed in terms of degrees of freedom on the bonds of the lattice, our system can be mapped onto a tensor network [11] as follows. Each of the N lattice sites has coordination number q and each bond can be in one of d possible states. Then the real-valued, symmetric tensor $T_{i_1 i_2 \cdots i_q}$ associated with a given site,

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doi:10.1016/j.physa.2010.02.010

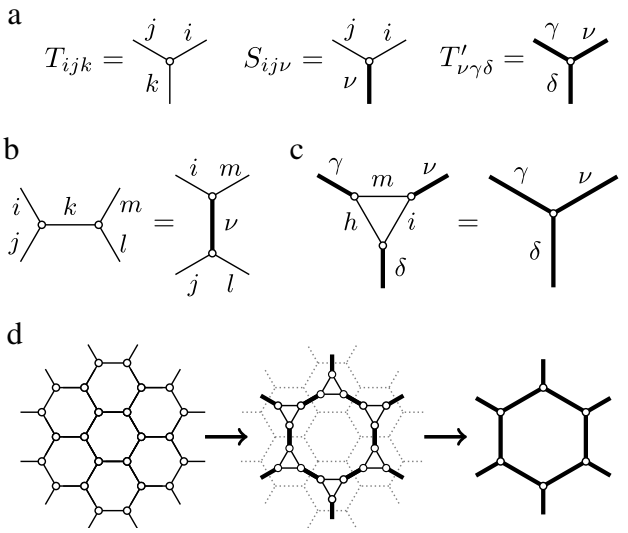

Fig. 1. (a) Graphical representation of the tensors. The convention is that the order of indices in the tensor matches the counterclockwise ordering of the labels on the vertex legs. (b) Rewiring. (c) Decimation. (d) RG transformation applied to the entire hexagonal lattice, with the first arrow showing the rewiring step, and the second arrow showing the decimation.
Source: (Reproduced from Ref. [8].)

where each index i_α runs from 1 to d, is a Boltzmann weight depending on the configuration of the q bonds meeting at the site. The partition function Z is a contraction over the N site tensors,

$$Z = \sum_{i_1,\ldots,i_M=1}^{d} T_{i_1 i_2 \cdots i_q} T_{i_1 i_r \cdots i_s} T_{i_2 i_t \cdots i_u} \cdots, \tag{1}$$

with each index contracted between two different tensors (since each bond is shared between two sites). Though the TRG method can be applied to various 2D geometries (i.e. the square and kagomé lattices [1]), it is simplest to describe it for a hexagonal lattice. Here we can graphically represent the tensors T_{ijk} as three-legged vertices (Fig. 1(a) left), and a contraction of an index between two tensors as joining two vertex legs. Thus Z as a hexagonal tensor network is shown on the left of Fig. 1(d), and the TRG transformation consists of a mapping of the original system onto a tensor network on a coarse-grained hexagonal lattice with $N' = N/3$ vertices (Fig. 1(d) right).

The mapping is carried out in two steps, known as rewiring and decimation. In the rewiring step the bonds of two neighboring tensors are reconnected as in Fig. 1(b), rewriting them as a contraction of two new tensors S,

$$\sum_{k=1}^{d} T_{ijk} T_{klm} = \sum_{\nu=1}^{d^2} S_{mi\nu} S_{jl\nu}. \tag{2}$$

While the m and i indices in the tensor $S_{mi\nu}$ run up to d, the ν index runs up to d^2 (graphically distinguished as a thick bond, as in Fig. 1(a) center). To find the S tensor, note that Eq. (2) can be written as a $d^2 \times d^2$ matrix equation, $M = SS^T$, where $M_{\alpha\beta} \equiv \sum_k T_{ijk} T_{klm}$, $S_{\alpha\nu} \equiv S_{mi\nu}$, and we introduce the composite indices $\alpha \equiv (m, i)$, $\beta \equiv (j, l)$. Since M is symmetric (due to the cyclical symmetry of T), singular value decomposition leads to the factorization [12]: $M = U\Sigma U^T$, where U is unitary and Σ is a diagonal matrix containing the d^2 singular values of M. Then the elements of S are given by $S_{\alpha\nu} = \sqrt{\Sigma_{\nu\nu}} U_{\alpha\nu}$, where $\Sigma_{\nu\nu}$ is the νth singular value (assumed ordered from largest to smallest with increasing ν). The rewiring procedure is applied to all pairs of T tensors in the lattice, as shown by the first arrow in Fig. 1(d), leading to a martini lattice involving only S tensors.

The decimation step in the TRG procedure traces over the degrees of freedom in the triangular clusters that are formed after the rewiring, replacing each triangle by a renormalized tensor T' (second arrow in Fig. 1(d)),

$$\sum_{m,i,h=1}^{d} S_{hm\gamma} S_{mi\nu} S_{ih\delta} = T'_{\gamma\delta\nu}. \tag{3}$$

Thus the complete mapping will express Z exactly in terms of a T' tensor network, though the tensor elements are now complex in general (due to the unitary matrix U that enters into the S construction). The tensor structure is also not preserved, because the index range of the tensors increases after the mapping, from d to d^2. Thus iteration of the transformation will lead to arbitrarily complicated tensors, rendering it numerically impractical. To make the TRG feasible, the index range is truncated by an upper bound D: instead of using the full matrix $S_{\alpha\nu}$ to calculate T', we use only the first

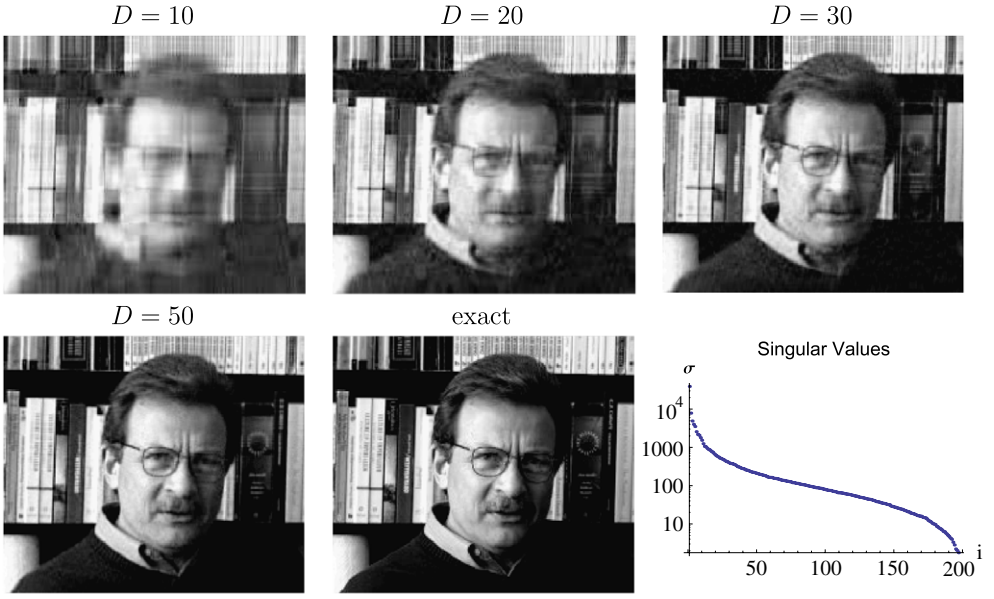

Fig. 2. An image compressed using singular value decomposition with a successively increasing cutoff, i.e. taking into account the largest $D = 10, 20, 30, 50$ singular values. The bottom center image is the exact original for comparison. Bottom right: a plot of the singular values of the original sorted by decreasing magnitude.

$\bar{d}$ columns, where $\bar{d} \equiv \min(d^2, D)$. In terms of this truncated matrix $\bar{S}$, the rewiring step becomes approximate, $M \approx \bar{S}\bar{S}^T$. Since the first $\bar{d}$ columns of S correspond to the $\bar{d}$ largest singular values, this approximation works surprisingly well even for small values of D, and can be systematically improved by increasing D. Using $\bar{S}$ instead of S in Eq. (3) means the indices of T' run up to $\bar{d}$, and the iterated TRG corresponds to flows in a finite-dimensional space of complex tensor elements.

The approximation in the rewiring step can be visualized through an analogy to image processing, where it is common to use singular value decomposition to compress images [13]. In Fig. 2, we see an image (a matrix M of pixel shadings), and successive approximations involving the D largest singular values. Even with D as small as 10 the main qualitative features are captured, and the approximation converges to the original as D increases.

To demonstrate the power of the TRG, in [1,8], the method was applied to an Ising model on a triangular lattice. Through a duality transformation, the partition function of this system can be rewritten as a tensor network with

$$T_{ijk} = \mathrm{e}^{\frac{\beta J}{2}(\sigma_i+\sigma_j+\sigma_k)} P(\sigma_i, \sigma_j, \sigma_k), \tag{4}$$

where J is the Ising nearest-neighbor coupling, $\beta = 1/k_B T$, $\sigma_i = -1, 1$ for $i = 1, 2$, and P is a projection operator, $P(\sigma_i, \sigma_j, \sigma_k) = (\sigma_i\sigma_j\sigma_k + 1)/2$.

As was shown in [8], the tools of traditional RG theory can be adapted to extract thermodynamic properties from the TRG flows. If we focus on the tensor element amplitudes, $|T_{\alpha\beta\gamma}|$, these flows are in a $D(2 + D^2)/3$-dimensional space (the number of distinct elements in a cyclically symmetric tensor with index range D). The low and high temperature phases in the ferromagnetic Ising model correspond to basins of attraction in this parameter space, associated not with isolated sinks, but rather continuous surfaces of fixed points. Despite this complication, critical temperatures and exponents can be deduced from properties of the boundaries between the basins, just like in a conventional RG. What distinguishes the TRG is the accuracy of the derived thermodynamic results: already at $D = 4$ the free energy is within 0.09% of the exact value at all temperatures, while the critical temperature and thermal eigenvalue are off by 3% and 2%, respectively. By $D = 24$, these error values have decreased to 0.0007% for the free energy, 0.02% for the critical temperature, and 0.9% for the thermal eigenvalue.

3. The TRG for disordered systems

Given the success of the TRG for pure systems, it is natural to extend the method to deal with more challenging aspects like quenched randomness [10]. Here the major difficulty is the spatial heterogeneity of the tensors in the original network, which leads to correlations between nearby tensors in the renormalized system that have to be accounted for in the rewiring and decimation steps. While these correlations can be treated approximately (updating a random pool of tensors at each step, along the lines of the Nobre method [15]), it is instructive to test the method first in a disordered system keeping all spatial correlations intact. To do this, we looked at large finite triangular lattices, with periodic boundary conditions, and a bond-diluted ferromagnetic Ising Hamiltonian with couplings J_{ij} distributed with probability $\mathcal{P}(J_{ij}) = p\delta(J_{ij}) + (1 - p)\delta(J_{ij} - J)$. The rewiring/decimation can be applied in a straightforward manner to all tensors in the network derived from the dual

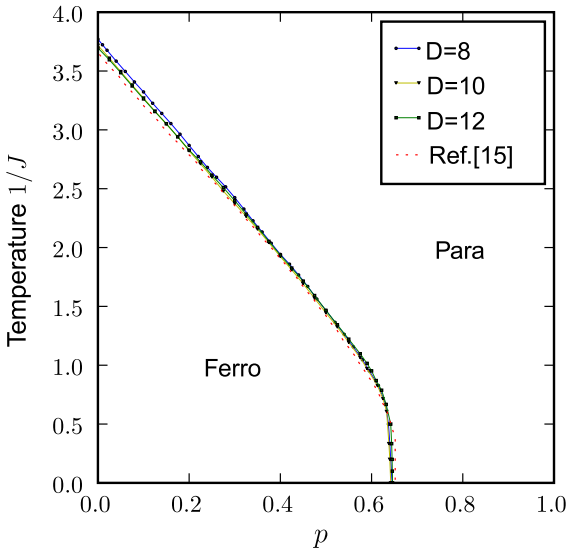

Fig. 3. The phase diagram of the bond-diluted triangular lattice Ising ferromagnet in terms of bond probability p versus temperature [10]. The boundary between the paramagnetic and ferromagnetic phases is calculated using the TRG for different cutoff parameters D and various system sizes N. The $D = 8$ and $D = 10$ results are for $N = 236\,196$ sites, while $D = 12$ is for $N = 78\,732$. The analytical approximation from [14] is drawn for comparison.

transform of the system, and the TRG is iterated until only a small number of renormalized spins are left, whose partition function can then be summed exactly.

The phase diagram of the bond-diluted Ising ferromagnet on a triangular lattice derived from the TRG is shown in Fig. 3 in terms of probability p versus temperature. For cutoffs $D = 8$–12 and system sizes $N \approx 8\times10^4$–2.4×10^5, the results converge to a smooth phase boundary, which agrees quite well with the analytical approximation of [14]. For zero temperature, where the boundary is purely a percolation transition, the exact threshold probability p_c is recovered with about 1% error [16].

4. Conclusion

In summary, the TRG is a versatile position space renormalization group technique for a wide range of classical 2D lattice Hamiltonians. While maintaining the general features of conventional RG methods, it offers substantive gains in accuracy, and is capable of systematic improvement through the cutoff parameter D. With the initial results outlined here, we show that it is possible to extend the method to incorporate quenched randomness, opening new avenues for future investigations, like spin glasses and other frustrated systems. We are currently working on a more thorough analysis of the random TRG method, including local external fields, comparing the effectiveness of approximations like the Nobre procedure [15], and designing computational enhancements to deal with larger systems or more complex Hamiltonians.

Acknowledgements

On the occasion of his 60th birthday, we would like to thank Nihat Berker for suggesting this problem, and for his strong encouragement and support throughout our academic development. We admire his enthusiasm, patience and trust in his students, and hope we can live up to his example by applying the intuition we gained from him in our future endeavors. We are also grateful to the Feza Gürsey Institute in Istanbul for its hospitality and support, particularly through the computational resources provided by the Gilgamesh cluster.

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