Odd $q$-state clock spin-glass models in three dimensions, asymmetric phase diagrams, and multiple algebraically ordered phases

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(Received 17 July 2014; published 4 December 2014)

Distinctive orderings and phase diagram structures are found, from renormalization-group theory, for odd $q$-state clock spin-glass models in $d = 3$ dimensions. These models exhibit asymmetric phase diagrams, as is also the case for quantum Heisenberg spin-glass models. No finite-temperature spin-glass phase occurs. For all odd $q \geq 5$, algebraically ordered antiferromagnetic phases occur. One such phase is dominant and occurs for all $q \geq 5$. Other such phases occupy small low-temperature portions of the phase diagrams and occur for $5 \leq q \leq 15$.

All algebraically ordered phases have the same structure, determined by an attractive finite-temperature sink fixed point where a dominant and a subdominant pair states have the only nonzero Boltzmann weights. The phase transition critical exponents quickly saturate to the high $q$ value.

I. INTRODUCTION

Spin-glass problems [1] continue to fascinate with new orderings and phase diagrams under frustration [2] and ground-state entropy [3,4]. The extension of these models from the extensively studied Ising spin models to less simple spins offers the possibility of completely new orderings and phase diagrams. We find that odd $q$-state clock models are such cases. Spins in odd $q$-state clock models cannot be exactly antialigned with each other. Furthermore, for a given spin, its interacting neighbor has two states that give the maximally misaligned pair configuration. This fact immediately injects ground-state entropy in the presence of antiferromagnetic interactions, even without the frozen randomness of interactions of the spin-glass system.

We have calculated, from renormalization-group theory, the phase diagrams of arbitrary odd $q$-state clock spin-glass models in $d = 3$ dimensions. We find that these models have asymmetric phase diagrams, as is also the case for quantum Heisenberg spin-glass models [5]. They exhibit no finite-temperature spin-glass phase. For all odd $q \geq 5$, algebraically ordered antiferromagnetic phases occur. One such phase is dominant and occurs for all $q \geq 5$. Other such phases occupy a small low-temperature portion of the phase diagram and occur for $5 \leq q \leq 15$. All algebraically ordered phases have the same structure, determined by an attractive finite-temperature sink fixed point where a dominant and a subdominant pair states have the only nonzero Boltzmann weights. The phase transition critical exponents quickly saturate to the high $q$ value.

II. THE ODD $q$-STATE CLOCK SPIN-Glass MODEL AND THE RENORMALIZATION-GROUP METHOD

The $q$-state clock models are composed of unit spins that are confined to a plane and that can only point along $q$ angularly equidistant directions. Accordingly, the $q$-state clock spin-glass model is defined by the Hamiltonian

$$-\beta H = \sum_{\langle ij \rangle} J_{ij} \vec{s}_i \cdot \vec{s}_j = \sum_{\langle ij \rangle} J_{ij} \cos(\theta_i - \theta_j),$$

where $\beta = 1/k_B T$, at site $i$ the spin angle $\theta_i$ takes on the values $(2\pi/q)\sigma_i$ with $\sigma_i = 0, 1, 2, \ldots, q - 1$, and $(\langle ij \rangle)$ denotes that the sum runs over all nearest-neighbor pairs of sites. The bond strengths $J_{ij}$ are $+J > 0$ (ferromagnetic) with probability $1 - p$ and $-J$ (antiferromagnetic) with probability $p$. This model becomes the Ising model for $q = 2$ and the XY model for $q \to \infty$.

The $q$-state clock spin-glass model, in $d = 3$ dimensions, is readily solved by a renormalization-group method that is approximate on the cubic lattice [6,7] and simultaneously exact on the hierarchical lattice [8–12]. Hierarchical lattices have been used to study a variety of spin-glass and other statistical mechanics problems [13–42]. Under rescaling, for $q > 4$, the form of the interaction as given in the rightmost side of Eq. (1) is not conserved and one must therefore express the Hamiltonian more generally, as

$$-\beta H = \sum_{\langle ij \rangle} V(\theta_i - \theta_j).$$

The energy $V(\theta_i - \theta_j)$ depends on the absolute value of the angle difference, $|\theta_i - \theta_j|$. Thus, the renormalization-group flows are the flows of $q/2$ interaction constants for even $q$ and the flows of $(q - 1)/2$ interaction constants for odd $q$. With no loss of generality, the maximum value of $V(\theta_i - \theta_j)$ is set to zero.

The renormalization-group transformation, for spatial dimensions $d = 3$ and length rescaling factor $b = 3$ (necessary for treating the ferromagnetic and antiferromagnetic correlations on equal footing), is achieved by a sequence of bond moving

$$V_{\text{fin}}(\theta_1 - \theta_2) + G_{12} = \sum_{\alpha=1}^{p_2-1} V_{\alpha}(\theta_1 - \theta_2)$$

DOI: 10.1103/PhysRevE.90.062112 PACS number(s): 64.60.De, 75.10.Nr, 05.10.Cc, 75.50.Lk

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shown in Fig. 2. The lower-temperature details of the phase diagrams are given in Fig. 3. All phase boundaries are second glass models are quite different from the even q

\[ q = 3, 5, 7, 9, 11, 13, 15, 17, 21, 361 \text{-state clock spin-glass models are shown in Fig. 2. The lower-temperature details of the phase diagrams are given in Fig. 3. All phase boundaries are second order.}

The phase diagrams of the odd q-state clock spin-glass models are quite different from the even q phase diagrams [21]: The odd q phase diagrams do not have ferromagnetic-antiferromagnetic symmetry, i.e., they are not left-right symmetric with respect to the \( p = 0.5 \) line. The odd q phase diagrams do not have a spin-glass phase, which is consistent with previous results [13, 21] that the XY model, corresponding to the \( q \to \infty \) limit of the q-state clock models, does not have a spin-glass phase on \( d = 3 \) hierarchical lattices. The odd q phase diagrams show a multiplicity of algebraically ordered phases (and one conventionally ordered phase) on the antiferromagnetic side. All points in an algebraically ordered phase flow, under renormalization group, to a single stable fixed point (sink) that occurs at nonzero, noninfinite temperature. Convergence to this stable critical fixed occurs, to six significant figures, within five renormalization-group transformations. Further convergence is obtained for more renormalization-group transformations. As seen in Fig. 4, at each renormalization-group transformation, the quenched probability distribution of interactions changes from the initial (1 - p) and p double-δ function, to eventually reach the critical sink described below. Because of this flow structure, the correlation length is infinite and the correlation function decays as an inverse power of distance (as opposed to exponentially)
FIG. 2. (Color online) Calculated phase diagrams of the odd $q$-state clock spin-glass models on the hierarchical lattice with $d = 3$ dimensions. These phase diagrams do not have ferromagnetic-antiferromagnetic symmetry, i.e., they are not left-right symmetric with respect to the $p = 0.5$ line. The phase diagrams do not have a spin-glass phase, but show a multiplicity of algebraically ordered phases on the antiferromagnetic side. The phase diagrams show true reentrance (disordered-ordered-disordered) as temperature is lowered at fixed antiferromagnetic bond concentration $p$, on both the ferromagnetic and antiferromagnetic sides of the phase diagram. The phase diagrams also show lateral, true double reentrance (ferromagnetic-disordered-ferromagnetic-disordered) as the antiferromagnetic bond concentration $p$ is increased at fixed temperature, only on the ferromagnetic side. No antiferromagnetic ordering occurs for the lowest model, $q = 3$. Algebraically ordered antiferromagnetic phases occur for all higher $q \geq 5$ models. In these cases, the phase boundary between the dominant antiferromagnetic algebraically ordered phase and the disordered phase is slightly asymmetric with the phase boundary between the ferromagnetic and disordered phases. To make this slight asymmetry evident, the latter boundary is also shown (dashed) reflected about the $p = 0.5$ line. The lower temperature details of these phase diagrams are shown in Fig. 3.
at all points in such an algebraically ordered phase. Such algebraically ordered phases were previously seen by Berker and Kadanoff [3,4] for antiferromagnetic Potts models and have since been extensively studied [44–52]. The correlation function decay critical exponent has the same value for all points in such a phase, since the renormalization-group flows are to single fixed point, in contrast to the continuously varying critical exponents in the algebraically ordered phase of the $d = 2$ XY model, where the flows are to a fixed line [53–55].

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The phase boundary between the ferromagnetic phase and the disordered phase. To make this slight asymmetry evident, the latter boundary is also shown (dashed) in Fig. 2 reflected about $p = 0.5$ line. The phase diagram for the $XY$ model limit, namely odd $q \to \infty$, is also shown in Fig. 2, calculated here with $q = 361$ clock states. In this limit, the distinction between odd and even $q$ disappears. This suggests that the zero-temperature spin-glass phase [60] found for even $q \to \infty$ [21] also occurs for odd $q \to \infty$.

IV. ALGEBRAICALLY ORDERED PHASES, FINITE-TEMPERATURE RENORMALIZATION-GROUP SINKS, AND GROUND-STATE ENTROPY

Spins in odd $q$-state clock models cannot be exactly antialigned with each other, i.e., $\theta_i - \theta_j = 2\pi q_{ij}/q < \pi$, where $q_{ij}$ is an integer between 0 and $(q-1)/2$ inclusive. Furthermore, for a given spin, its interacting neighbor has two states that give the maximally misaligned pair configuration with $\theta_i - \theta_j = \pi(q-1)/q < \pi$. Thus, for antiferromagnetic interaction, this local degeneracy is of crucial distinctive importance, injecting ground-state entropy into the system, driving the sink of a would-be ordered phase to nonzero temperature, and thereby causing algebraic order, as generally explained in Refs. [3,4].

All points in the antiferromagnetic phases in the phase diagrams in Figs. 2 and 3 flow under renormalization-group to $p = 1$ (just as all points in the one ferromagnetic phase flow to $p = 0$). The most extant antiferromagnetic phase in Fig. 2, labeled AF$_1$(alg), occurring for all odd $q \geq 5$ values, is an algebraically ordered phase. All points in this phase flow to a completely stable fixed point (a phase sink [61]) that is also a critical point since it occurs at finite temperature [3,4]. Of the pair-interaction Boltzmann weights $e^{V(6-q)/q}$, with $\theta_i - \theta_j = \pi(q-1-2n)/q$, where $n = 0$ is the most misaligned pair state, $n = 1$ is the next-most misaligned state, etc., until $n = (q-1)/2$ is the completely aligned pair state, only two are nonzero at this sink: The most misaligned pair state, $n = 0$, is dominant with $e^{V(\pi(q-1)/q)} = 1$ and the next-most misaligned pair state, $n = 1$, is also present but less dominant with $e^{V(\pi(q-3)/q)} = 1/3$. The other, less misaligned pair states, with $n \geq 2$, and the aligned pair state have zero Boltzmann weight at this sink. That these sink fixed-point Boltzmann weights are applicable for all odd $q$ is consistent with the fact that the $q-5$ less-misaligned pair states and one aligned pair state have negligible Boltzmann weights at the sink fixed point, so that the numerosity of $q$ does not matter. The finite difference between the energies for $\theta_i - \theta_j = \pi(q-1)/q$ and $\theta_i - \theta_j = \pi(q-3)/q$ establishes this sink as a finite-temperature attractive critical fixed point. It can be shown that, in the basin of attraction of a finite-temperature fixed point, the order parameter is strictly zero, the correlation length is infinite, and the correlations vanish algebraically with distance [3,4,11,62].

The evolution of the quenched probability distribution, under successive renormalization-group transformations, towards such a critical sink is shown in Fig. 4. The case of $q = 9$, starting with the initial condition temperature $1/J = 4$ and antiferromagnetic bond concentration $p = 0.8$ is shown in Fig. 4. For $q = 9$, the generalized interaction potential unavoidably generated by the renormalization-group transformation is determined by four interaction constants (see Table I). The renormalization-group transformation gives the evolution, under scale change, of the correlated quenched probability distribution $P(V_0, V_1, V_2, V_3, V_4)$ shown in this figure are the projections $P(V_1) = \int dV_2 dV_3 dV_4 P(V_0, V_1, V_2, V_3, V_4)$ and similarly for $P(V_2), P(V_3),$ and $P(V_4)$. Each row corresponds to another renormalization-group step $k$, as marked on the figure. It is seen that in four renormalization-group transformations, the renormalized system essentially reaches the critical phase sink described in Sec. IV: The most misaligned pair state is dominant with Boltzmann weight $e^{V(6\pi/9)} = 1$ and the next-most misaligned pair state is also present but less dominant with $e^{V(6\pi/9)} = 1/3$. The other two less misaligned pair states and the aligned pair state have zero Boltzmann weight at the sink.
ANTIFERROMAGNETIC PHASE-TRANSITION CRITICAL FIXED-POINT POTENTIALS $V(q-1)/q$, CRITICAL EXponents $y_T$, AND CORRESPONDING RELEVANT EIGENVECTORS OF DIFFERENT oDD q-STATE CLOCK MODELS. 

<table>
<thead>
<tr>
<th>$V(\theta_{ij})$</th>
<th>$n = 0$</th>
<th>$n = 1$</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
<th>$n = 4$</th>
<th>$n = 5$</th>
<th>$n = 6$</th>
<th>$y_T$</th>
<th>RELEVANT EIGENVECTORS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q = 5$</td>
<td>0</td>
<td>-0.0905</td>
<td>-0.1502</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.869030</td>
<td>1, 0.588</td>
</tr>
<tr>
<td>$q = 7$</td>
<td>0</td>
<td>-0.0538</td>
<td>-0.1242</td>
<td>-0.1569</td>
<td></td>
<td></td>
<td></td>
<td>0.873691</td>
<td>1, 0.782, 0.330</td>
</tr>
<tr>
<td>$q = 9$</td>
<td>0</td>
<td>-0.0345</td>
<td>-0.0893</td>
<td>-0.1395</td>
<td>-0.1599</td>
<td></td>
<td></td>
<td>0.873709</td>
<td>1, 0.866, 0.544, 0.206</td>
</tr>
<tr>
<td>$q = 11$</td>
<td>0</td>
<td>-0.0238</td>
<td>-0.0649</td>
<td>-0.1111</td>
<td>-0.1475</td>
<td>-0.1614</td>
<td></td>
<td>0.873709</td>
<td>1, 0.909, 0.675, 0.387, 0.140</td>
</tr>
<tr>
<td>$q = 13$</td>
<td>0</td>
<td>-0.0173</td>
<td>-0.0486</td>
<td>-0.0873</td>
<td>-0.1249</td>
<td>-0.1523</td>
<td>-0.1623</td>
<td>0.873709</td>
<td>1, 0.935, 0.759, 0.523, 0.287, 0.101</td>
</tr>
</tbody>
</table>

AF$_3$(alg), AF$_4$(alg), AF$_5$(alg), AF$_6$(alg), AF$_7$(alg), these roles are played respectively by $n = 2, 0$, $n = 1, 2, n = 2, 1, n = 1, 4$, $n = 4, 2$. On the other hand, AF$_3$(con) is a conventionally ordered phase, with a strong-coupling sink fixed point where $n = 1$ and $n = 4$ are equally dominant.

It is thus seen that the stable sink fixed points that attract, under renormalization-group flows, and characterize the algebraically ordered phases have identical structure for all odd $q \geq 5$. A similar, but not identical, phenomenon occurs for the unstable critical fixed points that control the antiferromagnetic phase transitions. This is seen in Fig. 5 and Table I, where the ferromagnetic ($p = 0$) and antiferromagnetic ($p = 1$) critical temperatures $1/J_C$ are given as a function of $q$. The fixed-point Boltzmann weights $e^{V(q-1)/2q}$ underpinning the antiferromagnetic phase transitions, as well as the critical exponents $y_T$ and corresponding relevant eigenvectors are given for different $q$ in Table I. For each $q$, the relevant eigenvector is the (only) relevant eigenvector of the $[(q-1)/2] \times [(q-1)/2]$ recursion matrix between the independent $V(\theta_{ij})$. Although the fixed points and relevant eigenvectors are distinct for different $q$, the critical temperatures and critical exponents quickly converge for high $q$, to $1/J_C = 12.2373$ and $y_T = 0.8737$. The critical temperatures and exponents thus show differences for low $q$. The convergence for high $q$ of the critical temperatures at $p = 0$ and $p = 1$ is expected, since the $q$-state clock models approach the $XY$ model for large $q$, with identical antiferromagnetic and ferromagnetic behavior.

V. CONCLUSION

We have calculated, from renormalization-group theory, the phase diagrams of arbitrary odd $q$-state clock spin-glass models in $d = 3$. These models have asymmetric phase diagrams, as is also the case for quantum Heisenberg spin-glass models [5]. For all odd $q \geq 5$, algebraically ordered antiferromagnetic phases occur. One such phase is dominant and occurs for all $q \geq 5$. Other such phases occupy small low-temperature portions of the phase diagrams and occur for $5 \leq q \leq 15$. All algebraically ordered phases have the same structure, determined by an attractive finite-temperature sink fixed point where a dominant and a subdominant pair states are the nonzero Boltzmann weights. The phase transition critical exponents, on the other hand, vary with $q$ only at low $q$.

A rich and distinctive phase transition structure is thus seen for odd $q$-state spin-glass models on a $d = 3$ dimensional hierarchical lattice.

ACKNOWLEDGMENTS

Support by the Alexander von Humboldt Foundation, the Scientific and Technological Research Council of Turkey (TÜBİTAK), and the Academy of Sciences of Turkey (TÜBA) is gratefully acknowledged.