$$p_{nm}(t) = \left[e^{\Omega t}\right]_{nm} = P_{nm}$$

prob. to end up at n at time t, given start at m at time 0

$$\frac{dP}{dt} = \Omega P \qquad \frac{dp_{nm}}{dt} = \sum_{m'} \Omega_{nm'} p_{m'm} = \sum_{m'} \left[W_{nm'} p_{m'm} - W_{m'n} p_{nm}\right]$$

$$\frac{dP}{dt} = P\Omega \qquad \frac{dp_{nm}}{dt} = \sum_{m'} p_{nm'} \Omega_{m'm}$$

$$= \sum_{m' \neq m} p_{nm'} \Omega_{m'm} + p_{nm} \Omega_{mm}$$

$$= \sum_{m' \neq m} p_{nm'} W_{m'm} - p_{nm} \sum_{m' \neq m} \overset{W}{\cancel{\Omega}}_{m'm}$$

$$\Rightarrow \boxed{\frac{dp_{nm}}{dt} = \sum_{m'} (p_{nm'} - p_{nm}) W_{m'm}}$$

adjoint ME

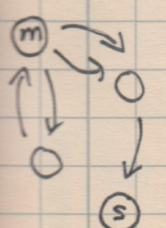
Survival probabilities + first passage times

Imagine network w/ sink (never can leave that state), or equivalently you are interested in chance that you first visit a state. Call that state s. ("sink")

$U_{sm}(t)$ = prob. that you haven't visited s by time t, given you started at m at $t=0$ $= \sum_{n \neq s} p_{nm}(t)$

$U_{ss}(t) = 0$ by definition

survival probability



~~take sum $\sum_{n \neq s}$ of both sides of adjoint ME:~~

$$\cancel{\frac{dU_{sm}}{dt} = \sum_{m'} \left[U_{sm'} - U_{sm}\right] W_{m'm}}$$

$$= \sum_{m'}{}' U_{sm'} W_{m'm} + U_{sm}\left[-\sum_{m'}{}' W_{m'm}\right]$$

Take $\sum_{n \neq s}$ of both sides of adjoint ME:

$$\sum_{n\neq s} \frac{d p_{nm}}{dt} = \sum_{n \neq s} \sum_{m'} p_{nm'} \Omega_{m'm}$$

$$\Rightarrow \quad \frac{dU_{sm}}{dt} = \sum_{m'} U_{sm'} \Omega_{m'm}$$

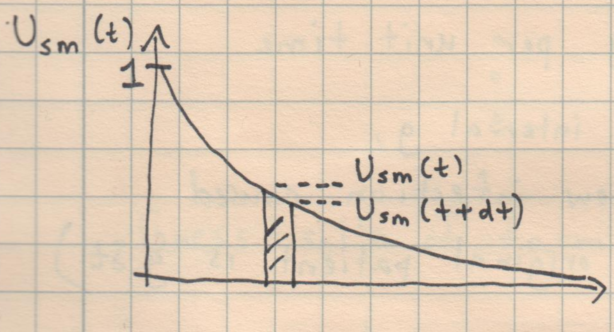
$$(*) \qquad = \sum_{m' \neq s} U_{sm'} \Omega_{m'm} \quad \text{b/c} \quad U_{ss} = 0$$

Note: if $m \neq s$ (don't start at sink)

$$U_{sm}(0) = 1$$

+ as $t \to \infty$ if you have one sink + network otherwise strongly connected

$$U_{sm}(t \to \infty) = 0$$



$$U_{sm}(t) - U_{sm}(t+dt) = -\frac{dU_{sm}}{dt} dt$$

= fraction of trajectories that reach s between t + t+dt

$$\equiv dt \; f_{sm}(t)$$

↑ distribution of first passage times (first arrival at s)

mean first passage time (MFPT):

$$\tau_{sm} = \int_0^\infty dt \, f_{sm}(t) \, t$$

$$= -\int_0^\infty dt \frac{dU_{sm}}{dt} t = \cancelto{0}{-U_{sm} t \Big|_0^\infty} + \int_0^\infty dt \, U_{sm}$$

cool result: $\tau_{sm} = \int_0^\infty dt\, U_{sm}$ $\quad \tau_{ss} = 0$

$\uparrow$ MFPT from m to s

Integrate both sides from 0 to ∞ of Eq. * for $m \neq s$:

$$\cancelto{0}{U_{sm}}(\infty) - \cancelto{1}{U_{sm}}(0) = \sum_{m' \neq s} \tau_{sm'} \Omega_{m'm}$$

$$\Rightarrow \boxed{\begin{array}{c} -1 = \sum_{m' \neq s} \tau_{sm'} \Omega_{m'm} \quad (m \neq s) \\ \tau_{ss} = 0 \end{array}}$$

system of equations to calc. τ_{sm} for all m

shortened notation: τ_m instead of τ_{sm} (s implicit)

Example: a model for disease spreading

I) very simple: one infected individual, A spreads disease to new person w/ probability g per unit time (during every time interval g, the prob. of a new infection caused directly by our original patient is $g\,\delta t$)

At time t, what is the prob. that the orig. individual has infected n others? (ignore secondary infections)

states: $⓪ \xrightarrow{g} ① \xrightarrow{g} ② \xrightarrow{g} ③ \xrightarrow{g} \cdots$

# of infections caused by indiv.

"Poisson process"

$$\begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\ \Omega = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} & \begin{bmatrix} -g & & & & \\ g & -g & & & \\ & g & -g & & \\ & & g & \ddots & \\ & & & & \ddots \end{bmatrix} \end{array}$$

master equation: $\frac{dp_n}{dt} = \sum_m \Omega_{nm} p_m$ <u>or</u> $\frac{dp_{n0}}{dt} = \sum_m \Omega_{nm} p_{m0}$

equivalent

$$\Rightarrow \quad \frac{dp_0}{dt} = -g\,p_0 \Rightarrow p_0 = e^{-gt}$$

$$\frac{dp_1}{dt} = g\,p_0 - g\,p_1$$

$$\frac{dp_2}{dt} = g\,p_1 - g\,p_0$$

$$\vdots$$

Guess sol'n of form: $p_n(t) = c_n(t)\,e^{-gt}$ $\quad c_0(t) = 1$

$$\frac{dp_n}{dt} = g\,p_{n-1} - g\,p_n$$

$$\Rightarrow \dot{c}_n e^{-gt} \cancel{- g c_n e^{-gt}} = g\,c_{n-1} e^{-gt} \cancel{- g c_n e^{-gt}}$$

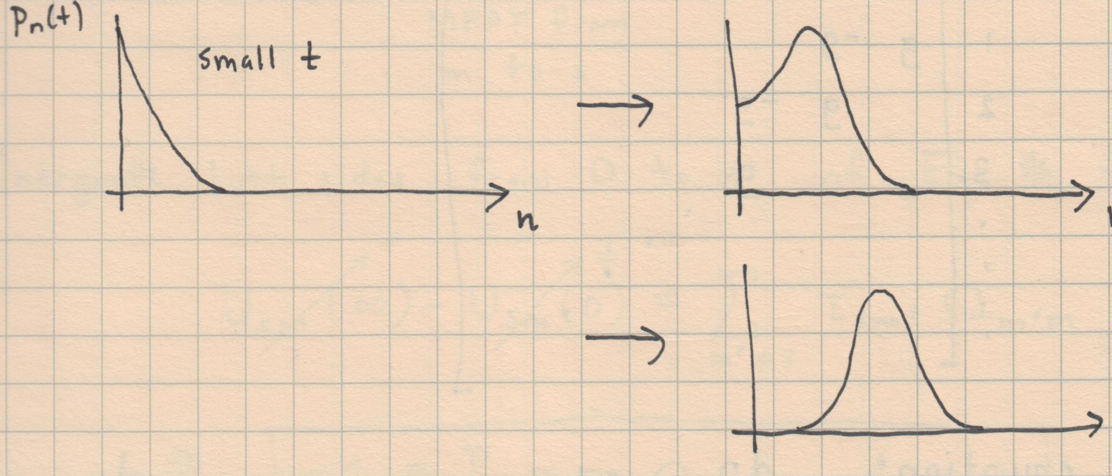
$$\Rightarrow \dot{c}_n = g\,c_{n-1} \quad \text{w/} \quad c_0(t) = 1$$

$$\dot{c}_1 = g \Rightarrow c_1 = gt$$

$$\dot{c}_2 = g^2 t \Rightarrow c_2 = \frac{(gt)^2}{2}$$

$$\Rightarrow c_n = \frac{(gt)^n}{n!}$$

$$p_n(t) = \frac{(gt)^n}{n!} e^{-gt}$$ Poisson distribution



peak moves to right as t increases

mean n: $$\langle n \rangle_t = \sum_{n=0}^{\infty} n\, p_n(t)$$

$$= \sum_{n=1}^{\infty} n \frac{(gt)^n}{n!} e^{-gt}$$ b/c no term at $n=0$

$$= \sum_{n=1}^{\infty} \frac{(gt)^n}{(n-1)!} e^{-gt}$$

$$= gt \sum_{n=1}^{\infty} \frac{(gt)^{n-1}}{(n-1)!} e^{-gt}$$

$$= gt \underbrace{\sum_{m=0}^{\infty} \frac{(gt)^m}{m!} e^{-gt}}_{= \sum_{m=0}^{\infty} p_m(t) = 1}$$ $m = n-1$

$$= gt$$ mean increases w/ t

II) Now imagine our infected individual returns to being susceptible ($I \to S$) w/ probability r ~~to~~ per unit time. On average how many infections will this individual cause during the duration of the ~~inf~~ illness.

model for individual A: (ignore infections caused)

$$I \xrightarrow{r} S$$

$$\Omega = \begin{array}{c c} & \begin{array}{cc} I & S \end{array} \\ \begin{array}{c} I \\ S \end{array} & \begin{pmatrix} -r & 0 \\ r & 0 \end{pmatrix} \end{array}$$

$U_{SI}(t)$ = "survival prob." of illness
= prob. the individual is still ~~in I~~ not in S at time t

$$\frac{dU_{SI}}{dt} = \sum_{m' \neq S} U_{Sm'} \Omega_{m'm} = U_{SI}\Omega_{II} = -r\,U_{SI}$$

$$\Rightarrow \quad U_{SI}(t) = e^{-rt} \quad \text{w/} \quad U_{SI}(0) = 1$$

first passage (to S) time distrib:

$$f_{SI}(t) = -\frac{dU_{SI}}{dt} = re^{-rt}$$

MFPT τ_{SI} : $\quad -1 = \sum_{m' \neq S} \tau_{Sm'} \Omega_{m' I}$

$$\Rightarrow \quad -1 = \tau_{SI}\Omega_{II} \quad \Rightarrow \quad -1 = \tau_{SI}(-r)$$

$$\tau_{SI} = \frac{1}{r}$$ mean length of illness

prob. to infect n individuals ~~given~~ during illness

$$\mathcal{P}(n) = \int_0^\infty dt \underbrace{\frac{(gt)^n}{n!} e^{-gt}}_{\text{prob. to infect } n \text{ during time } t} \cdot \underbrace{re^{-rt}}_{\text{prob. illness lasts time } t}$$

using identity: $\int_0^\infty t^n e^{-at} = \frac{n!}{a^{n+1}}$ $\Rightarrow$ $\mathcal{P}(n) = \frac{rg^n}{(r+g)^{n+1}}$

$a>0$, integer $n \geq 0$

$$= \frac{\left(\frac{g}{r}\right)^n}{\left(1+\frac{g}{r}\right)^{n+1}}$$

mean # of cases caused by individual during illness:

$$\sum_{n=0}^\infty n\,\mathcal{P}(n) = \sum_{n=0}^\infty \frac{nrg^n}{(r+g)^{n+1}} = \frac{r}{r+g} \sum_{n=0}^\infty n \left(\frac{g}{r+g}\right)^n$$

identity: $\sum_{n=0}^\infty n x^n = \frac{x}{(1-x)^2}$ $\Rightarrow$ $= \frac{r}{r+g} \frac{\frac{g}{r+g}}{\left(1-\frac{g}{r+g}\right)^2} = \frac{g}{r}$

$|x|<1$

This mean # is called $\boxed{R_0 = \frac{g}{r}}$ basic reproduction number of disease

common cold: $R_0 \approx 6$

measles: $R_0 \approx 15$

Ebola: $R_0 \approx 2$

coronavirus: $R_0 \approx 2\text{-}3$?

salmonella: $R_0 < 1$

seasonal flu: $R_0 = 1.4$

III) Finite population of N people

state n represents n people currently ill.

transition rates: $n \to n-1$ w/ rate $r_n = nr$ for $n \geq 1$

(any of n people can recover)

$n \to n+1$ w/ rate $g_n = ng$ for $n \geq 1$

(any of n people can pass their illness to a new person)

$$\begin{tikzcd} N \arrow[r, shift left, "r_N"] & N-1 \arrow[l, shift left, "g_{N-1}"] \arrow[r, shift left, "r_{N-1}"] & N-2 \arrow[l, shift left, "g_{N-2}"] & \cdots & 1 \arrow[r, shift left, "r_1"] & 0 \arrow[l, shift left, "g_0"] \end{tikzcd}$$

g_0 = rate at which disease is reintroduced from "wild" in a healthy population (might be very small)

This network is ergodic, hence should have a unique stationary state p_n^s.

$p_n(t) \longrightarrow p_n^s$ as $t \to \infty$

What is p_n^s?

$$\Omega = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\ \begin{pmatrix} -g_0 & r_1 & & \\ g_0 & -r_1 - g_1 & r_2 & \\ & g_1 & -r_2 - g_2 & r_3 \\ & & g_2 & \ddots \end{pmatrix} \end{array}$$

$$\frac{dp_n}{dt} = \sum_m \Omega_{nm} p_m$$

$$= g_{n-1} p_{n-1} - (r_n + g_n) p_n + r_{n+1} p_{n+1}$$

$N > n \geq 1$

for $p_n^s \Rightarrow \frac{dp_n}{dt} = 0$

$$0 = -g_0 p_0^s + r_1 p_1^s$$
$$0 = +g_0 p_0^s - (r_1 + g_1) p_1^s + r_2 p_2^s$$
$$0 = g_1 p_1^s - (r_2 + g_2) p_2^s + r_3 p_3^s$$

$$\to \quad p_1^s = \frac{g_0}{r_1} p_0^s$$

plug into next equ: $p_2^s = \frac{g_1}{r_2} p_1^s = \frac{g_1 g_0}{r_2 r_1} p_0^s$

+ so on: $p_n^s = \frac{g_{n-1} \cdots g_0}{r_n \cdots r_1} p_0^s$ $\qquad g_i = i g$, $r_i = i r$

$$= \frac{g_0}{nr} \left(\frac{g}{r}\right)^{n-1} p_0^s$$

$$= \frac{g_0}{ng} \left(\frac{g}{r}\right)^{n} p_0^s = \frac{g_0}{g} \frac{R_0^n}{n} p_0^s$$

normalize: $1 = p_0^s + \sum_{n=1}^{N} p_n^s = p_0^s \left[1 + \frac{g_0}{g} \sum_{n=1}^{N} \frac{R_0^n}{n} \right]$

for $N \to \infty$: $\sum_{n=1} \frac{R_0^n}{n} \to -\ln(1 - R_0)$ if $R_0 < 1$

check by Taylor series

In this limit: $p_0^s = \frac{1}{1 - \frac{g_0}{g} \ln(1 - R_0)} = \begin{cases} 1 \text{ if } g_0 = 0 \\ \to 0 \text{ as } R_0 \to 1 \end{cases}$

prob. of no one sick as $t \to \infty$

for $R_0 \geq 1$: $\sum_{n=1}^{N} \frac{R_0^N}{n} \to \infty$ as $N \to \infty$: $p_0^s = 0$

~~blows up like $\log(N) + \gamma$~~

Euler's const ≈ 0.577

goal to ~~prevent~~ keep disease under control: get $R_0 < 1$