We need entropy dynamics to resolve this question. So far we have defined:

$$\dot{E}(t) = \frac{d}{dt} \sum_n p_n(t) E_n = \sum_{(n,m)} J_{nm}(t) (E_n - E_m)$$

= energy flow

Since $E_n - E_m = -(E_q - E_{q'})$ for $(m,q') \rightarrow (n,q)$ transition

$$\dot{E}(t) = -\dot{E}_{\text{env}}(t) \quad \text{where} \quad \dot{E}(t) + \dot{E}_{\text{env}} = E_{\text{tot}} = \text{const}$$

positive energy flow into system \quad \text{negative energy flow out of environment}

\{ \text{note:} \quad \dot{E}(t) \text{ can be } >0 \text{ or } <0 \text{ or } 0 \}

General entropy dynamics (valid for any system obeying master equation w/ a stat. state)

$$\dot{S}(t) = -k_B \sum_n \dot{p}_n \ln \frac{p_n}{\bar{p}_n} - k_B \sum_n \dot{p}_n \frac{d}{dt} \sum_n \dot{p}_n = \frac{d}{dt} 1 = 0$$

= $-k_B \sum_{n,m} J_{nm} \ln p_n$ using $\dot{p}_n = \sum_m J_{nm}$

= $-k_B \sum_{(n,m)} J_{nm} (\ln p_n - \ln p_m)$

= $-k_B \sum_{(n,m)} J_{nm} \ln \left( \frac{W_{mn} p_n}{W_{nm} p_m} \right) + k_B \sum_{(n,m)} J_{nm} \ln \left( \frac{W_{mn} p_n}{W_{nm} p_m} \right)$

= $\dot{S}^i(t) + \dot{S}^e(t)$
Properties: (universally valid)

i) \( \dot{S}^i(t) \geq 0 \) at all times

we will call this later: "internal entropy production"

Proof: 
\[
\dot{S}^i(t) = -k_B \sum_{(n,m)} (W_{nm} p_m - W_{nm} p_n) \ln \frac{W_{nm} p_n}{W_{nm} p_m}
\]

\[
= k_B \sum_{n,m} W_{nm} p_m \ln \frac{W_{nm} p_m}{W_{nm} p_n}
\]

define prob. vectors \( \vec{\rho} \) and \( \vec{\rho}' \)
by: 
\[
\rho_{nm} = \frac{W_{nm} p_m}{C} \quad \text{where} \quad C = \sum_{n,m} W_{n,m} p_m
\]
\[
\rho'_{nm} = \frac{W_{nm} p_n}{C}
\]

\[ \implies \dot{S}^i(t) = k_B C \cdot D_{KL}(\vec{\rho} \parallel \vec{\rho}') \geq 0 \text{ always} \]

corollary: \( \dot{S}^i(t) = 0 \) iff \( \rho_{nm} = \rho'_{nm} \) for all \( n,m \)

\[ \implies W_{nm} p_m = W_{nm} p_n \]
\[ \implies J_{nm} = 0 \text{ for all } n,m \]
detailed balance is satisfied (only can occur at \( t = \infty \))

Any nonzero current \( J_{nm}(t) \) will make \( \dot{S}^i(t) > 0 \)

ii) \( \dot{S}^e(t) = k_B \sum_{(nm)} J_{nm}(t) \ln \frac{W_{nm}}{W_{nm}} \) \( \Rightarrow \) later identified as "entropy flow" from environment

\( \dot{S}^e(t) \) can be positive or negative:

i.e. 
\[
\begin{array}{c}
\text{m} \\
\downarrow \text{W}_{mn} \\
\text{n}
\end{array}
\]

If \( W_{mn} > W_{nm} \) but \( p_m(t) \gg p_n(t) \) \( \Rightarrow \dot{S}^e(t) > 0 \)

\[ \implies \ln \frac{W_{nm}}{W_{nm}} > 0 \implies J_{nm}(t) > 0 \]
If \( W_{mn} \geq W_{nm} \Rightarrow p_m(t) \ll p_n(t) \Rightarrow \dot{S}^e \leq 0 \)
\[ \Rightarrow \ln \frac{W_{mn}}{W_{nm}} > 0 \Rightarrow J_{nm}(t) < 0 \]

Hence \( \dot{S}(t) = \dot{S}^i(t) + \dot{S}^e(t) \), \( \dot{S}^i(t) \geq 0 \)
at \( t \to \infty \), \( \dot{S}(t) = -k_B \sum_n \dot{p}_n \ln p_n = 0 \) since \( \dot{p}_n = 0 \)
\[ \Rightarrow 0 = \dot{S}^i(\infty) + \dot{S}^e(\infty) \equiv \dot{S}^i,s + \dot{S}^e,s \]

If ESS (all \( J_{nm}^s = 0 \): \( \dot{S}^i,s = 0 \), \( \dot{S}^e,s = 0 \)
If NESS (some \( J_{nm}^s \neq 0 \): \( \dot{S}^i,s > 0 \), \( \dot{S}^e,s = -\dot{S}^i,s \leq 0 \)

So far this is general. Let us focus on system at temp. \( T \) connected to heat bath.

In this special case we have shown:
\[ D_{KL}(\hat{p}(t) \| \hat{p}^s) = \frac{1}{k_B T} (F(t) - F^s) \geq 0 \]
\[ \frac{d}{dt} D_{KL} = 0 \Rightarrow \frac{\dot{F}(t)}{k_B T} = 0 \Rightarrow \dot{F}(t) = 0 \]
\[ \Rightarrow \text{Since } F(t) = \bar{E}(t) - TS(t) \]
\[ S(t) = -\frac{F(t)}{T} + \frac{\bar{E}(t)}{T} \]
\[ \dot{S}(t) = -\frac{\dot{F}(t)}{T} + \frac{\dot{\bar{E}}(t)}{T} \]
\[ \geq 0 \text{ always can be posit., or} \]
\[ = 0 \text{ only at } t = \infty \text{ negative} \]
Compare to: $\dot{S}(t) = \dot{S}^i(t) + \dot{S}^e(t)$

Matching terms:

\[
\begin{aligned}
\dot{S}^i(t) &= -\frac{\dot{F}(t)}{T} \\
\dot{S}^e(t) &= \frac{\dot{E}(t)}{T}
\end{aligned}
\]

true for the case of heat bath at temp $T$

so $\dot{S}^e$ corresponds to classical (adiabatic) definition of entropy:

\[dS = \frac{dE}{T}\]

(since $dQ = dE$ when no work)

$\dot{S}^i$ is the extra entropy produced away from stationary state.

Note we have an explicit formula for $\dot{S}^i(t)$,

\[
\dot{S}^i(t) = -k_B \sum_{n,m} J_{n,m} \ln \frac{W_{nm} P_n}{W_{nm} P_m}
\]

unlike in traditional thermodynamics.

In this case when $t \to \infty$, $\dot{F}(t) \to 0$

$\dot{S}^i(t) \to 0$

\[\Rightarrow J_{nm} = 0 \text{ for all } n,m\]

*** Hence for the heat bath at temp $T$,

the stationary state of system must be ESS

$\Rightarrow$ no currents, detailed balance is satisfied.

Amazingly remarkable restriction: any system at temp $T$ w/ heat bath must have transition matrix satisfying

\[
\begin{bmatrix}
\frac{W_{nm}}{W_{mn}} \\
\frac{P_n}{P_m}
\end{bmatrix} = e^{-\beta(E_n - E_m)}
\]
Since $W$ is time-indep., ratio is valid at all times.

Finally, remember that

$$S^{\text{tot}}(t) = S(t) + \sum_p P_n S^{\text{envin}}$$

$$\Rightarrow \dot{S}^{\text{tot}}(t) = \dot{S}(t) - \frac{\dot{E}(t)}{T} = -\frac{\dot{F}(t)}{T} = \dot{S}^i(t)$$

Hence internal entropy production = rate at which the total entropy of environment increases

$\dot{S}^e$ does not contribute, since any entropy flow into system is compensated for by entropy flow out of environment.