PHYS 414: 4-24-20

$$\hat{\rho} = \sum_n p_n |\psi_n\rangle\langle\psi_n|$$

$\downarrow$ measuring some observ. $\hat{A}$

$$\hat{\rho}' = \sum_a' p_a |a\rangle\langle a|$$

$$p_a = \sum_n' p_n |\langle a|\psi_n\rangle|^2$$

$$\hat{\rho}' = \sum_{a,n} p_n \langle a|\psi_n\rangle\langle\psi_n|a\rangle |a\rangle\langle a|$$

$$= \sum_a \underbrace{|a\rangle\langle a|}_{\hat{P}_a} \left[\sum_n' p_n |\psi_n\rangle\langle\psi_n|\right] \underbrace{|a\rangle\langle a|}_{\hat{P}_a}$$

proj. operator corresp. to $|a\rangle$

$$P_a^\dagger = |a\rangle\langle a| = P_a$$

$$\boxed{\hat{\rho}' = \sum_a \hat{P}_a \hat{\rho} \hat{P}_a^\dagger}$$

dens. oper. changing under measurement

$$\boxed{\hat{\rho}' = \hat{U}\hat{\rho}\hat{U}^\dagger}$$

dens. operator changing for an isolated quant. sys.

example: imagine an initial pure ensemble 100% in state $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$\hat{\rho} = |\psi_1\rangle\langle\psi_1|$$

$$S(\hat{\rho}) = -\sum_n p_n \ln p_n$$

$\hookrightarrow$ probs in the ON decomposition $\Big\}$ basis where $\hat{\rho}$ is diag.

here ON basis is $|\psi_1\rangle$, $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

matrix of $\hat{\rho}$ in this basis: $\hat{\rho} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$p_1 = 1$, p_2

$$\Rightarrow S(\hat{\rho}) = 0$$

Do a measurement that projects us onto the $|0\rangle, |1\rangle$ basis w/ equal probability

after meas: $\hat{\rho}' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

ON basis is : $|0\rangle, |1\rangle$

$\hat{\rho}' = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{2}$

$$S(\hat{\rho}') = \ln 2 > S(\hat{\rho}) = 0$$

In general, $S(\hat{\rho}') \geq S(\hat{\rho})$ for measurements.

Can we think more generally of a quantum system interacting w/ environment, of which projective measurements are one special case?

$\Rightarrow$ Theory of open quantum systems?

Central theorem: the most general transf. that can occur for a dens. oper. of sys interacting w/ environment looks like:

$$\hat{\rho}' = \sum_{\gamma=1}^{\Gamma} \hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^\dagger \qquad \Gamma \le N^2$$

where N is dim. of Hilbert space of sys.

for some operators $\hat{M}_\gamma$ that satisfy

$$\sum_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma = \hat{\mathbb{1}} \quad \text{identity}$$

- Choi - Kraus representation theorem
- operator - sum representation
- quantum channel repr.

previous examples all fall under this general form:

i) isolated sys: $\hat{\rho}' = U \hat{\rho} U^\dagger \qquad \Gamma = 1$

$$\hat{M}_1 = \hat{U}$$

$$\sum_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma = \hat{U}^\dagger \hat{U} = \hat{\mathbb{1}}$$

ii) measurement: $\hat{\rho}' = \sum_a \hat{P}_a \hat{\rho} \hat{P}_a^\dagger \qquad \hat{P}_a = |a\rangle\langle a|$

$$\Gamma = N$$

$$\hat{M}_\gamma \Rightarrow |a\rangle\langle a|$$

$$\sum_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma = \sum_a |a\rangle\underbrace{\langle a|a\rangle}_{1}\langle a| = \sum_a |a\rangle\langle a| = \hat{\mathbb{1}}$$

Proof: how can we take one valid dens. oper $\hat{\rho}$ and map it onto another valid dens. oper $\hat{\rho}'$?

valid: $\hat{\rho}^\dagger = \hat{\rho}$, $\operatorname{tr}(\hat{\rho}) = 1$, $\langle i|\hat{\rho}|i\rangle \ge 0$ in any basis also demand that our mapping be <u>linear</u>:

for example

$$\hat{\rho}_1 = |\psi_1\rangle\langle\psi_1| \rightsquigarrow \hat{\rho}_1'$$

$$\hat{\rho}_2 = |\psi_2\rangle\langle\psi_2| \rightsquigarrow \hat{\rho}_2'$$

$$\hat{\rho} = p\,\hat{\rho}_1 + (1-p)\,\hat{\rho}_2 \rightsquigarrow \hat{\rho}' = p\,\hat{\rho}_1' + (1-p)\,\hat{\rho}_2'$$

this is the correct dens. oper. after mapping

choose some basis of our Hilbert space of dim. N

$$\{|i\rangle\}$$

$$\Rightarrow \quad \hat{\rho} = \sum_{ij} \rho_{ij}\,|i\rangle\langle j| \qquad \hat{\rho}' = \sum_{\ell k} \rho'_{\ell k}\,|\ell\rangle\langle k|$$

$$\rho_{ij} = \langle i|\hat{\rho}|j\rangle \qquad \rho'_{\ell k} = \langle \ell|\hat{\rho}'|k\rangle$$

linearity $\Rightarrow$ describe relation b/t $\hat{\rho}'$ + $\hat{\rho}$ using a matrix $\Rightarrow$ $\rho'_{\ell k} = \sum_{ij} \underbrace{S_{\ell i;kj}}\,\rho_{ij}$

for a given ℓ, k this is a matrix that tells us how much ρ_{ij} contributes to $\rho'_{\ell k}$

Define composite variables:

$$\alpha \equiv (\ell, i) \quad [N^2 \text{ dim.}]$$

$$\beta \equiv (k, j)$$

$S_{\ell i;k\ell} \Rightarrow S_{\alpha\beta}$ where S is an $N^2 \times N^2$ matrix

$\hat{\rho}' = \sum_{\ell,k} \rho'_{\ell k}\,|\ell\rangle\langle k|$ plus $\rho'_{\ell k} = \sum_{ij} S_{\ell i;kj}\,\rho_{ij}$ and $\rho_{ij} = \langle i|\hat{\rho}|j\rangle$

$$\Rightarrow \quad \hat{\rho}' = \sum_{\alpha,\beta} S_{\alpha\beta}\,\langle i|\hat{\rho}|j\rangle\,|\ell\rangle\langle k|$$

for any $\gamma = (m, n) \Rightarrow \hat{\tau}_\gamma = |m\rangle\langle n|$

$\hat{\tau}_\gamma^\dagger = |n\rangle\langle m|$

$$\hat{\rho}' = \sum_{\alpha\beta} S_{\alpha\beta} |l\rangle\langle i| \hat{\rho} |j\rangle\langle k|$$

$$= \sum_{\alpha\beta} S_{\alpha\beta} \hat{\tau}_\alpha \hat{\rho} \hat{\tau}_\beta^\dagger$$