PHYS 414: 4-27-20

continuing w/ derivation of Choi-Krauss repr. theorem:

- for any linear mapping that takes one quant. operator $\hat{\rho} \longrightarrow \hat{\rho}'$ we found

$$\hat{\rho}' = \sum_{\beta,\alpha} S_{\alpha\beta}\, \hat{\tau}_\alpha \hat{\rho} \hat{\tau}_\beta^\dagger \qquad \begin{array}{l} \alpha = (\ell, i) \\ \beta = (k, j) \\ \quad N^2 \end{array} \qquad \{|i\rangle\}$$

compl. basis of dim. N

$$\hat{\tau}_\alpha = |\ell\rangle\langle i|$$
$$\hat{\tau}_\beta = |k\rangle\langle j|$$

- now we will restrict this mapping to only those that preserve the properties of density matrices
  $\Rightarrow$ put constraints on $S_{\alpha\beta}$

i) $\hat{\rho}'^\dagger = \hat{\rho}'$ (Hermitian)

$$\hat{\rho}'^\dagger = \sum_{\alpha\beta} S^*_{\alpha\beta}\, \hat{\tau}_\beta \hat{\rho} \hat{\tau}_\alpha^\dagger \underset{\substack{\alpha\to\beta \\ \beta\to\alpha}}{=} \sum_{\alpha\beta} S^*_{\beta\alpha}\, \hat{\tau}_\alpha \hat{\rho} \hat{\tau}_\beta^\dagger \overset{?}{=} \hat{\rho}'$$

$$\Rightarrow \boxed{S^*_{\beta\alpha} = S_{\alpha\beta}} \Rightarrow S \text{ must be a Hermitian matrix}$$

$\exists$ some unitary matrix U that diag. S:

$$\underbrace{U^\dagger S U}_{\text{columns are e-vecs of } S} = \underbrace{\Lambda}_{\text{diag elem are e-vals of } S} \Rightarrow S = U \Lambda U^\dagger$$

$$S_{\alpha\beta} = \sum_{\gamma,\mu} U_{\alpha\gamma} \underbrace{\Lambda_{\gamma\mu}}_{\lambda_\gamma \delta_{\gamma\mu}} \underbrace{(U^\dagger)_{\mu\beta}}_{U^*_{\beta\mu}}$$

$\hookrightarrow$ e-val of S

$$= \sum_\gamma U_{\alpha\gamma} \lambda_\gamma U^*_{\beta\gamma}$$

now let's plug into our mapping expression

$$\Rightarrow \hat{\rho}' = \sum_{\alpha\beta\gamma} \lambda_\gamma U_{\alpha\gamma} \hat{\tau}_\alpha \hat{\rho} \hat{\tau}^\dagger_\beta U^*_{\beta\gamma}$$

$$= \sum_\gamma \epsilon_\gamma \hat{M}_\gamma \hat{\rho} \hat{M}^\dagger_\gamma \qquad \epsilon_\gamma = \text{sign}(\lambda_\gamma) = \pm 1$$

real b/c S is Herm.

where
$$\hat{M}_\gamma \equiv \sum_\alpha U_{\alpha\gamma} \hat{\tau}_\alpha \sqrt{|\lambda_\gamma|} \rightsquigarrow \text{Kraus matrices}$$

$$\hat{M}^\dagger_\gamma = \sum_\alpha U^*_{\alpha\gamma} \hat{\tau}^\dagger_\alpha \sqrt{|\lambda_\gamma|} = \sum_\beta U^*_{\beta\gamma} \hat{\tau}^\dagger_\beta \sqrt{|\lambda_\gamma|}$$

(there can be up to N^2 of them)

γ runs to N^2

ii) $\text{tr}(\hat{\rho}') = 1 \Rightarrow \sum_\gamma \epsilon_\gamma \, \text{tr}(\hat{M}_\gamma \hat{\rho} \hat{M}^\dagger_\gamma) = 1$

$$= \sum_\gamma \epsilon_\gamma \, \text{tr}(\hat{M}^\dagger_\gamma \hat{M}_\gamma \hat{\rho})$$

$\text{tr}(\hat{A}\hat{B}\hat{C}) = \text{tr}(\hat{C}\hat{A}\hat{B})$

$$= \text{tr}\Big(\underbrace{\Big[\sum_\gamma \epsilon_\gamma \hat{M}^\dagger_\gamma \hat{M}_\gamma\Big]}_{\hat{A}} \hat{\rho} \Big)$$

note: Hermitian

can choose a basis where $\hat{A}$ is diagonal

$\Rightarrow$ call that basis $\{|m\rangle\}$

$$= \sum_m \langle m|\hat{A}\hat{\rho}|m\rangle = \sum_{m,n} \underbrace{\langle m|\hat{A}|n\rangle}_{A_{mm}\delta_{mn}} \langle n|\hat{\rho}|m\rangle$$

$$= \sum_m A_{mm} \rho_{mm} = 1$$

also know $tr(\hat{\rho}) = \sum_m \rho_{mm} = 1$

the only way both can be true is if $A_{mm} = 1$ for all m

$$\Rightarrow \hat{A} = \hat{\mathbb{1}} = \sum_\gamma \epsilon_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma$$

iii) in any basis diag. elements of $\hat{\rho}'$ have to ≥ 0

$$\langle i|\hat{\rho}'|i\rangle = \sum_\gamma \epsilon_\gamma \langle i|\hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^\dagger|i\rangle \geq 0$$

must be valid for any $\hat{\rho}$, in particular

$\hat{\rho} = |\psi\rangle\langle\psi|$ (pure state)

$$\Rightarrow \langle i|\hat{\rho}'|i\rangle = \sum_\gamma \epsilon_\gamma \langle i|\hat{M}_\gamma|\psi\rangle\langle\psi|\hat{M}_\gamma^\dagger|i\rangle$$

$$= \sum_\gamma \epsilon_\gamma |\langle i|\hat{M}_\gamma|\psi\rangle|^2 \geq 0$$

for this to be true for any $|\psi\rangle$

$\Rightarrow$ we will add the constraint $\epsilon_\gamma = +1$ for all γ

$\Rightarrow$ further constrains S to only have non-negative e-vals $\Rightarrow$ S is Hermitian + positive semi-definite

$\Rightarrow$ Choi-Krauss theorem

$$\boxed{\hat{\rho}' = \sum_{\gamma=1}^{N^2} \hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^\dagger \quad \text{where} \quad \sum_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma = \hat{\mathbb{1}}}$$

$\hat{M}_\gamma$ = Krauss matrix

Goal: quantum master equation describing time evol of $\hat{\rho}(t)$

$$\frac{\partial}{\partial t}\hat{\rho}(t) = \sim\sim\sim$$

To get this we can imagine a system interacting w/ environment in general, where every time step we have:

$$\hat{\rho}(t+\delta t) = \sum_{\gamma} \hat{M}_{\gamma}\, \hat{\rho}(t)\, \hat{M}_{\gamma}^{\dagger}$$

implicitly assumed Markovian property

write this somehow

$$= \hat{\rho}(t) + \delta t\,(\sim\sim)$$