RG Methods in Statistical Field Theory: Problem Set 3 Solution

In class we discussed the spin wave fluctuations which occur when a continuous symmetry is broken. In this problem set, we will see that such fluctuations can actually destroy the ordered state completely under certain conditions. We will work in $d$ dimensions, and concentrate on the case of an order parameter with $n = 2$ components (known as the XY model).

Let us start at the same place we did in class: by taking the mean-field solution and adding small fluctuations to it. Assume the mean-field solution has the form $m(x) = m \hat{e}_1$, where $m$ is independent of $x$, and $\hat{e}_1$ is a unit vector along the direction in which the system orders at low temperature. Instead of using the $\phi_\parallel$ and $\phi_\perp$ fluctuations we introduced in class, we choose to write the fluctuations in a different form, more convenient when the $m(x)$ vector has only $n = 2$ components:

$$m(x) = m \cos \theta(x) \hat{e}_1 + m \sin \theta(x) \hat{e}_2$$

Here $\theta(x)$ is an angle that can vary with position. When there are no fluctuations, $\theta(x) = 0$, and we get the mean-field solution with all the $m(x)$ vectors pointing in the same direction.

(a) First, let us calculate the energy of the fluctuations. Plug the above form for $m(x)$ into the Hamiltonian functional:

$$H[m(x)] = \int d^d x \left[ \frac{r}{2} m^2(x) + um^4(x) + \frac{c}{2} \langle \nabla m(x) \rangle^2 \right]$$

Show that $H$ can be written as:

$$H = H_0 + \frac{K}{2} \int d^d x \langle \nabla \theta(x) \rangle^2$$

where $H_0 = V(\frac{r}{2} m^2 + um^4)$ is just the mean-field energy, and $K = cm^2$.

**Answer:** Let us plug $m(x) = m \cos \theta(x) \hat{e}_1 + m \sin \theta(x) \hat{e}_2$ into each term in the Hamiltonian:

$$m^2(x) = m(x) \cdot m(x) = m^2 \cos^2 \theta(x) + m^2 \sin^2 \theta(x) = m^2$$

$$m^4(x) = (m(x) \cdot m(x))^2 = m^4$$

$$\langle \nabla m(x) \rangle^2 = \sum_{i=1}^{2} \sum_{\alpha=1}^{d} \partial_\alpha m_i(x) \partial_\alpha m_i(x)$$

$$= m^2 \sin^2 \theta(x) \sum_{\alpha} (\partial_\alpha \theta(x))^2 + m^2 \cos^2 \theta(x) \sum_{\alpha} (\partial_\alpha \theta(x))^2$$

$$= m^2 \sum_{\alpha} (\partial_\alpha \theta(x))^2$$

Putting everything together, we find:

$$H = \int d^d x \left[ \frac{r}{2} m^2 + um^4 + \frac{cm^2}{2} \langle \nabla \theta(x) \rangle^2 \right]$$
\[ V \left( \frac{r m^2}{2} + u m^4 \right) + \int d^d x \frac{c m^2}{2} (\nabla \theta(x))^2 \]
\[ = H_0 + \int d^d x \frac{K}{2} (\nabla \theta(x))^2 \]

(b) Now imagine the system is a box of volume \( V = L^d \), and write \( \theta(x) \) as a Fourier expansion:

\[ \theta(x) = \frac{1}{V} \sum_q e^{i q \cdot x} \theta(q), \quad \theta(q) = \int d^d x e^{-i q \cdot x} \theta(x) \]

where \( q = \frac{2\pi}{L} (n_1 \mathbf{e}_1 + n_2 \mathbf{e}_2 + \ldots n_d \mathbf{e}_d) \) and the \( n_i \) are integers. The functions \( e^{i q \cdot x} \) satisfy the orthogonality condition:

\[ \int d^d x e^{i (q - q') \cdot x} = V \delta_{q,q'} \]

Show that the Hamiltonian can be written as:

\[ H = H_0 + \frac{K}{2V} \sum_q q^2 \theta(q) \theta(-q) \]

Answer:

\[ H = H_0 + \int d^d x \frac{K}{2} (\nabla \theta(x))^2 \]
\[ = H_0 - \frac{K}{2V^2} \int d^d x \sum_{q,q'} q \cdot q' e^{i (q + q') \cdot x} \theta(q) \theta(q') \]
\[ = H_0 - \frac{K}{2V^2} \sum_{q,q'} V \delta_{q,-q'} q \cdot q' \theta(q) \theta(q') \]
\[ = H_0 + \frac{K}{2V} \sum_q q^2 \theta(q) \theta(-q) \]

(c) Use the fact that \( \theta(x) \) is real to show that \( \theta(-q) = \theta^*(q) \). This means that \( \theta(q) \theta(-q) = \theta_R^2(q) + \theta_I^2(q) \), where \( \theta_R(q) \) and \( \theta_I(q) \) are the real and imaginary parts of \( \theta(q) \). Show that the Hamiltonian can be written as:

\[ H = H_0 + \frac{K}{V} \sum_{q>0} q^2 \left[ \theta_R^2(q) + \theta_I^2(q) \right] \]

Here the sum over \( q > 0 \) is shorthand notation that means we are summing over only half of the possible values of \( q \). (For example, we restrict one of the integers \( n_i \) to be positive.)

Answer:

\[ \theta^*(q) = \int d^d x e^{i q \cdot x} \theta(x) = \theta(-q) \]

Using \( \theta(q) \theta(-q) = \theta_R^2(q) + \theta_I^2(q) \) we can write the Hamiltonian as:

\[ H = H_0 + \frac{K}{2V} \sum_q q^2 \left[ \theta_R^2(q) + \theta_I^2(q) \right] \]
Note that $\theta_R(-q) = \theta_R(q)$ and $\theta_I(-q) = -\theta_I(q)$, so in the sum both $q$ and $-q$ contribute the same value $q^2 [\theta_R^2(q) + \theta_I^2(q)]$. Thus we can restrict the sum to half of $q$ space, and multiply it by a factor of 2:

$$\mathcal{H} = \mathcal{H}_0 + \frac{K}{V} \sum_{q > 0} q^2 [\theta_R^2(q) + \theta_I^2(q)]$$

(d) The partition function involves integrating over all possible functions $m(x)$. In terms of the Fourier-transformed Hamiltonian, this means integrating over all possible values of the Fourier components $\theta_R(q)$ and $\theta_I(q)$:

$$Z = \int_{-\infty}^{\infty} \prod_{q > 0} d\theta_R(q) d\theta_I(q) e^{-\beta \mathcal{H}}$$

We are interested in calculating the average of $m(x)$ along the $\hat{e}_1$ direction, which we can write as follows:

$$\langle m_1(x) \rangle = m \langle \cos \theta(x) \rangle = m \Re \langle e^{i\theta(x)} \rangle$$

where $\Re z$ denotes the real part of a complex number $z$. Thus to find $\langle m_1(x) \rangle$ we have to find the average:

$$\langle e^{i\theta(x)} \rangle = \frac{1}{Z} \int_{-\infty}^{\infty} \prod_{q > 0} d\theta_R(q) d\theta_I(q) e^{i\theta(x)} e^{-\beta \mathcal{H}}$$

Replace $\theta(x)$ by its Fourier expansion: it turns out that the integral above can be rewritten as a product over ordinary Gaussian integrals, which can be solved using the basic rule we showed in class:

$$\int_{-\infty}^{\infty} d\phi e^{-\frac{K}{2} \phi^2 + h\phi} = \sqrt{\frac{2\pi}{K}} e^{h^2/2K}$$

where $K$ and $h$ can be complex, with $\Re K > 0$. Show that:

$$\langle m_1(x) \rangle = me^{-W} \quad \text{where} \quad W = \frac{1}{\beta KV} \sum_{q > 0} \frac{1}{q^2}$$

Answer:

$$\langle e^{i\theta(x)} \rangle = \frac{1}{Z} \int_{-\infty}^{\infty} \prod_{q > 0} d\theta_R(q) d\theta_I(q) e^{i\theta(x)} e^{-\beta \mathcal{H}}$$

$$= \frac{1}{Z} \int_{-\infty}^{\infty} \prod_{q > 0} d\theta_R(q) d\theta_I(q) e^{\frac{1}{\beta} \sum_{q > 0} e^{i q \cdot x} \theta(q) e^{-\beta H_0 - \frac{\beta K}{V} \sum_{q > 0} q^2 [\theta_R^2(q) + \theta_I^2(q)]}}$$

$$= \frac{1}{Z} \int_{-\infty}^{\infty} \prod_{q > 0} d\theta_R(q) d\theta_I(q) e^{\frac{1}{\beta} \sum_{q > 0} (e^{i q \cdot x} \theta(q) + e^{-i q \cdot x} \theta(-q)) e^{-\beta H_0 - \frac{\beta K}{V} \sum_{q > 0} q^2 [\theta_R^2(q) + \theta_I^2(q)]}}$$

$$= \frac{e^{-\beta H_0}}{Z} \prod_{q > 0} \int_{-\infty}^{\infty} d\theta_R(q) d\theta_I(q) e^{-\frac{\beta K}{V} q^2 [\theta_R^2(q) + \theta_I^2(q)] + \frac{1}{\beta} (e^{i q \cdot x} \theta(q) + e^{-i q \cdot x} \theta(-q))}$$

$$= \frac{e^{-\beta H_0}}{Z} \prod_{q > 0} \int_{-\infty}^{\infty} d\theta_R(q) d\theta_I(q) e^{-\frac{\beta K}{V} q^2 [\theta_R^2(q) + \theta_I^2(q)] + \frac{1}{\beta} \cos(q \cdot x) \theta_R(q) - \sin(q \cdot x) \theta_I(q))}$$
\[ e^{-\beta H_0} = \frac{1}{Z} \prod_{\mathbf{q} > 0} e^{-\beta \mathbf{q}^2 V} e^{-\frac{\lambda^2\mathbf{q}^2}{\sqrt{\beta V}}} e^{-\beta \lambda^2\mathbf{q}^2} \]
\[ = \prod_{\mathbf{q} > 0} e^{-1/\langle V \beta \mathbf{q}^2 \rangle} \]
\[ = e^{-\frac{1}{\pi \mathbf{q}^2} \sum_{\mathbf{q} > 0} \frac{1}{\mathbf{q}^2}} \equiv e^{-W} \]

Clearly \( W \) is real. Thus:
\[ \langle m_1(\mathbf{x}) \rangle = m \Re \langle e^{i\theta(\mathbf{x})} \rangle = me^{-W} \]

(e) Mean-field theory tells us that the constant \( m \) will be nonzero below \( T_c \). If \( W < \infty \), then the result of part (d) shows us that we still have an ordered phase at low temperatures, though with an average magnetization \( \langle m_1(\mathbf{x}) \rangle = me^{-W} \) that is smaller than the mean-field solution because of the effects of fluctuations. However, if \( W = \infty \), we get the interesting result that \( \langle m_1(\mathbf{x}) \rangle = 0 \): the ordered phase has been destroyed by the fluctuations! Calculate \( W \), and show that there is an ordered phase for dimensions \( d > 2 \). For \( d \leq 2 \) show that there is no order except at \( T = 0 \).

Hint: So how do we calculate the value of \( W \)? In the limit of large volume we can replace the sum over \( \mathbf{q} \) by an integral:
\[ W = \frac{1}{\beta KV} \sum_{\mathbf{q} > 0} \frac{1}{\mathbf{q}^2} \rightarrow \frac{1}{2\beta K} \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{\mathbf{q}^2} \]

where we add the factor of 1/2 because we make the integral go over all of \( \mathbf{q} \)-space, not just one-half. We have to be careful here: when we expanded \( \theta(\mathbf{x}) \) in terms of Fourier components, we did not specify any restrictions on \( \mathbf{q} \). However it is unphysical to include fluctuations with such large \( |\mathbf{q}| \) that the wavelengths \( \lambda = 2\pi/|\mathbf{q}| \) are smaller than the microscopic lattice spacing of our system \( \ell \). Thus our integral should not really be over all \( \mathbf{q} \)-space, but rather within some cutoff \( |\mathbf{q}| < \Lambda \), where \( \Lambda \propto 1/\ell \). We are integrating inside a \( d \)-dimensional sphere of radius \( \Lambda \), where \( \Lambda \) is large but not infinite. With this restriction in place, we can now calculate the integral. For \( d > 1 \) the infinitesimal \( d \)-dimensional volume \( d^d\mathbf{q} \) can be written in radial coordinates as \( d^d\mathbf{q} = q^{d-1} dq d\Omega_d \), where \( d\Omega_d \) is a \( d \)-dimensional solid angle. The angular integration can be done using the fact that:
\[ \int d\Omega_d = S_d \quad \text{where} \quad S_d = \frac{2\pi^{d/2}}{(d/2 - 1)!} \]

Here \( S_d \) is the area of a \( d \)-dimensional unit sphere.

Answer: Writing \( W \) as an integral:
\[ W = \frac{1}{2\beta K} \int d^d\mathbf{q} \frac{1}{\mathbf{q}^2} \]
\[ = \frac{k_B T S_d}{2K(2\pi)^d} \int_0^\Lambda dq \frac{q^{d-1}}{q^2} \]
\[ = \frac{k_B T S_d}{2K(2\pi)^d} \int_0^\Lambda dq q^{d-3} \]
For $d \leq 2$ this integral blows up, so $W = \infty$ and there is no ordered phase for $T \neq 0$. When $d > 2$ the integral is convergent, and the expression for $W$ becomes:

$$W = \frac{k_B T S_d \Lambda^{d-2}}{2K(2\pi)^d(d-2)}$$

Thus there is an ordered phase at low temperatures, with the magnetization suppressed by a factor of $e^{-W}$ due to fluctuations.