# RG Methods in Statistical Field Theory: Problem Set 6

due: Friday, November 10, 2006

## Problem 1

In this problem we investigate the nature of the singularities in the Gaussian model as $T \to T_c^+$ ($r \to 0^+$). Even though at $r = 0$ the system exhibits fluctuations at all length scales, we will show that the singularities are caused entirely by long-wavelength fluctuations (small $\mathbf{q}$ modes).

**(a)** Consider the $d$-dimensional Gaussian model, written in terms of Fourier-transformed variables $\mathbf{m}(\mathbf{q})$, where $\mathbf{m}$ is the $n$-component order parameter:

$$\mathcal{H} = \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{2} \left( r + cq^2 + Lq^4 + \cdots \right) |\mathbf{m}(\mathbf{q})|^2 - \mathbf{H} \cdot \mathbf{m}(\mathbf{q} = 0)$$

Here $\mathbf{H} = H\hat{\mathbf{e}}_1$ is a uniform magnetic field pointing along the $\hat{\mathbf{e}}_1$ axis. Using the facts about Gaussian functional integrals discussed earlier in class, find the exact expression for the partition function $Z$ of this system. Show that the free energy per volume $f$ can be written as:

$$f = -\frac{1}{\beta V} \ln Z = \frac{n}{2\beta} \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \ln \left[ v_0^{-1} \beta (r + cq^2 + Lq^4 + \cdots) \right] - \frac{H^2}{2r}$$

*Hint:* Depending on how you calculate $Z$, you might end up with a factor of $\delta^{(d)}(\mathbf{q} = 0)$ in one of the terms. You can find the value of this factor using the definition: $(2\pi)^d \delta^{(d)}(\mathbf{q}) = \int dx \, \exp(i\mathbf{q} \cdot \mathbf{x})$. Thus $\delta^{(d)}(0) = V/(2\pi)^d$, where $V$ is the volume of the system.

**(b)** Let us look at the magnetic susceptibility, $\chi = -\partial^2 f / \partial H^2$ evaluated at $H = 0$. Show that $\chi \propto r^{-1}$, so it diverges as $r \to 0^+$. Note that this divergence is entirely due to the $\mathbf{H} \cdot \mathbf{m}(\mathbf{q} = 0)$ term in the Hamiltonian $\mathcal{H}$, where the magnetic field couples to the $\mathbf{q} = 0$ mode (infinite wavelength fluctuation). The singularity does not depend in any way on the cutoff $\Lambda$. If we change the cutoff, adding or subtracting high $\mathbf{q}$ modes in the Hamiltonian, the singular behavior of $\chi$ is not affected.

**(c)** Calculate the leading behavior of the specific heat for small $r$ at $H = 0$, $C \approx -T_c \partial^2 f / \partial r^2$. Show that it can be written as:

$$C \approx A \int_0^\Lambda dq \frac{q^{d-1}}{(r + cq^2 + Lq^4 + \cdots)^2}$$

where the constant $A = nk_B T_c^2 S_d / 2(2\pi)^d$ and $S_d$ is the area of a $d$-dimensional unit sphere. Argue that for $d > d_c$, there is no divergence in $C$ as $r \to 0^+$. Find $d_c$.

**(d)** Now consider the case $d < d_c$. Let us break up the integral into two parts, one going from $q = 0$ to $\Lambda/b$, and the other from $q = \Lambda/b$ to $\Lambda$:

$$C \approx A \int_0^{\Lambda/b} dq \frac{q^{d-1}}{(r + cq^2 + Lq^4 + \cdots)^2} + A \int_{\Lambda/b}^\Lambda dq \frac{q^{d-1}}{(r + cq^2 + Lq^4 + \cdots)^2} \equiv C_< + C_>$$

Argue that for any $b > 1$, the contribution $C_>$ must be finite in the limit $r \to 0^+$.

**(e)** The result of part (d) means that the divergence in $C$ is entirely contained in the $C_<$ term. Show that as $r \to 0^+$, $C_< \approx Br^{-\alpha}$, where $B$ is a constant independent of $\Lambda$ and $b$. Find the exponent $\alpha$. *Hint:* Non-dimensionalize the $C_<$ integral using the variable $x = (c/r)^{1/2}q$.

Note that parts (d) and (e) are true for any $b > 1$, even in the limit $b \gg \Lambda$, where $C_<$ corresponds to an integral over a tiny ball of radius $\Lambda/b$ surrounding $\mathbf{q} = 0$ in the Brillouin zone. Thus the small $\mathbf{q}$ modes determine the divergence in the specific heat. The cutoff $\Lambda$, or any other details of the high $\mathbf{q}$ behavior, have no affect on the singularity.

## Problem 2

Up to now we have only considered systems with short-range interactions. In magnetic lattice models we had a nearest-neighbor spin-spin interaction, and in the continuum limit this gave us derivative terms like $(\nabla \mathbf{m}(\mathbf{x}))^2$ in the Landau-Ginzburg Hamiltonian. But real physical systems can also have long-range effects, decaying slowly with distance, like magnetic dipole-dipole interactions. How would such interactions affect the critical behavior? In this problem we look at this question in the context of the Gaussian model.

**(a)** Let us add a long-range interaction $\mathcal{H}_{LD}$ to the Hamiltonian of the $d$-dimensional Gaussian model, where:

$$\mathcal{H}_{LD} = \int d^d\mathbf{x} \int d^d\mathbf{y} J(|\mathbf{x} - \mathbf{y}|)\mathbf{m}(\mathbf{x}) \cdot \mathbf{m}(\mathbf{y})$$

and $J(r) = A/r^{d+\sigma}$ for some constants $A, \sigma > 0$. Show that in terms of Fourier modes, this interaction can be written as:

$$\mathcal{H}_{LD} = K_\sigma \int \frac{d^d\mathbf{q}}{(2\pi)^d} q^\sigma \mathbf{m}(\mathbf{q}) \cdot \mathbf{m}(-\mathbf{q})$$

where $K_\sigma$ is a constant which depends on the value of $\sigma$. *Hint:* It is useful to change variables to $\mathbf{R} = (\mathbf{x} + \mathbf{y})/2$ and $\mathbf{r} = (\mathbf{x} - \mathbf{y})/2$. There will be an integral over $\mathbf{r}$ from which the $\mathbf{q}$ dependence can be factored out using the substitution $\mathbf{s} = q\mathbf{r}$. The constant $K_\sigma$ involves an integral (independent of $\mathbf{q}$) which you do *not* need to evaluate.

**(b)** Thus the Gaussian model with the long-range interaction has the form:

$$\mathcal{H} = \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{2} \left(r + K_\sigma q^\sigma + cq^2 + Lq^4 + \cdots\right) |\mathbf{m}(\mathbf{q})|^2 - \mathbf{H} \cdot \mathbf{m}(\mathbf{q} = 0)$$

Construct a renormalization-group transformation for this system, and find equations for $r'$, $K'_\sigma$, $c'$, $L'$, .... Leave the equations in terms of the parameter $\zeta$, where $\zeta$ is the constant of proportionality in the definition $\mathbf{m}'(\mathbf{q}') = \zeta^{-1}\mathbf{m}_<(\mathbf{q})$. (Do not choose a particular value for $\zeta$ just yet.)

**(c)** Consider the case where $\sigma > 2$, $c > 0$, and $K_\sigma, L, \ldots$ have arbitrary values. Choose an appropriate $\zeta$, and show that the long-range interaction is irrelevant at the fixed point: it does not affect the critical behavior of the system.

**(d)** Consider the case where $\sigma < 2$, $K_\sigma > 0$, and $c, L, \ldots$ have arbitrary values. Choose an appropriate $\zeta$, and calculate the critical exponents $\gamma$, $\nu$, and $\eta$. You should find that some of the exponents in this case depend on $\sigma$. Thus if the decay of the long-range interaction is sufficiently slow ($\sigma < 2$), it affects the critical behavior of the system.