Consider a one-dimensional lattice of sites $x_\alpha$, where the lattice spacing $x_{\alpha+1} - x_\alpha = \ell$. At each site $x_\alpha$ we have a quantity $\phi_\alpha$ that is a continuous variable ranging between $-\infty$ and $\infty$. The Hamiltonian for this system is:

$$
\mathcal{H} = \sum_\alpha \left[ A (\cosh(\phi_\alpha) - 1) + \frac{K}{2} (\phi_{\alpha+1} - \phi_\alpha)^2 \right],
$$

where the constants $A, K > 0$.

The partition function for this model is given by:

$$
Z = \int_{-\infty}^{\infty} \prod_\alpha d\phi_\alpha \exp (-\beta \mathcal{H}).
$$

In the continuum limit we can write each $\phi_\alpha$ as $\phi_\alpha = \phi(x_\alpha)$, where $\phi(x)$ is a continuous function of $x$. In this limit the partition function becomes (to lowest order) the functional integral:

$$
Z = \int \mathcal{D}\phi(x) \exp \left( -\beta \int dx \left[ \frac{r}{2} \phi^2(x) + u \phi^4(x) + \frac{c}{2} \left( \frac{\partial}{\partial x} \phi(x) \right)^2 + \cdots \right] \right).
$$

Find the coupling constants $r$, $u$, and $c$ in terms of $A$, $K$ and $\ell$. Hint: Write out $\phi(x_{\alpha+1})$ as a Taylor series around $\phi(x_\alpha)$. Also, use the fact that $\cosh(x) = 1 + x^2/2 + x^4/24 + \cdots$ for small $x$.

**Answer:** We expand $\phi(x_{\alpha+1}) = \phi(x_\alpha + \ell)$ as a Taylor series around $\phi(x_\alpha)$:

$$
\phi(x_\alpha + \ell) = \phi(x_\alpha) + \ell \frac{\partial}{\partial x} \phi(x_\alpha) + \cdots.
$$

Plugging this into the Hamiltonian, and using the Taylor expansion for $\cosh(x)$, we find:

$$
\mathcal{H} = \sum_\alpha \left[ \frac{A}{2} \phi^2(x_\alpha) + \frac{A}{24\ell} \phi^4(x_\alpha) + \frac{K\ell^2}{2} \left( \frac{\partial}{\partial x} \phi(x_\alpha) \right)^2 + \cdots \right].
$$

We pull a factor of $\ell$ outside the brackets:

$$
\mathcal{H} = \sum_\alpha \ell \left[ \frac{A}{2\ell} \phi^2(x_\alpha) + \frac{A}{24\ell} \phi^4(x_\alpha) + \frac{K\ell}{2} \left( \frac{\partial}{\partial x} \phi(x_\alpha) \right)^2 + \cdots \right].
$$

In the continuum limit $\sum_\alpha \ell \rightarrow \int dx$ and $\phi(x_\alpha) \rightarrow \phi(x)$, so we have:

$$
\mathcal{H} = \int dx \left[ \frac{r}{2} \phi^2(x) + u \phi^4(x) + \frac{c}{2} \left( \frac{\partial}{\partial x} \phi(x) \right)^2 + \cdots \right],
$$

where:

$$
r = \frac{A}{\ell}, \quad u = \frac{A}{24\ell}, \quad c = K\ell.
$$