RG Methods in Statistical Field Theory:  
Quiz 4 Solution  
Friday, October 20, 2006

Consider a $d$-dimensional system with an $n = 1$ component order parameter $m(x)$. There is a mean-field solution $m_0$, and fluctuations away from the mean-field solution $m(x) = m_0 + \phi(x)$ are described by the Fourier-transformed Hamiltonian:

$$H = H_0 + \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} c(q^2 + \xi^{-2}) \phi(q)\phi(-q)$$

Here $c > 0$ is a constant, and $\xi$ is the correlation length.

(a) We are interested in the correlation function for the fluctuations:

$$G(x, x') = \langle \phi(x)\phi(x') \rangle - \langle \phi(x) \rangle\langle \phi(x') \rangle$$

Assume the system is translationally invariant, so that $G(x, x') = G(x - x')$. What is the Fourier-transformed correlation function $G(q)$ for the Hamiltonian above? (No long calculations are necessary. You can write it down by inspection.)

**Answer:** The partition function for this system is:

$$Z = e^{-\beta H_0} \int \mathcal{D}m(q)e^{-\frac{1}{2} \int \frac{d^d q}{(2\pi)^d} c(q^2 + \xi^{-2}) \phi(q)\phi(-q)}$$

This has the Gaussian functional integral form discussed in class, with $K(q) = \beta c(q^2 + \xi^{-2})$. Thus the correlation function is given by:

$$G(q) = \frac{1}{K(q)} = \frac{1}{\beta c(q^2 + \xi^{-2})}$$

(b) Now do an inverse Fourier transform and write down an integral for the same-site correlation function $G(x, x)$. (Do not evaluate the integral.) This measures the magnitude of fluctuations at a site $x$. Show that for $d > 2$, we can approximate the integral and write

$$G(x, x) \approx \frac{S_d \Lambda^{d-2}}{\beta c(2\pi)^d (d-2)}$$

where $S_d$ is the area of a $d$-dimensional unit sphere, and $\Lambda$ is a large cutoff in $q$-space. **Hint:** Non-dimensionalize the integral using the variable $y \equiv q\xi$.

**Answer:**

$$G(x, x') = \int_0^\Lambda \frac{d^d q}{(2\pi)^d} G(q)e^{i\mathbf{q}\cdot(x-x')} = \int_0^\Lambda \frac{d^d q}{(2\pi)^d} \frac{e^{i\mathbf{q}\cdot(x-x')}}{\beta c(q^2 + \xi^{-2})}$$

$$\Rightarrow G(x, x) = \int_0^\Lambda \frac{d^d q}{(2\pi)^d} \frac{1}{\beta c(q^2 + \xi^{-2})} = \frac{S_d}{(2\pi)^d} \int_0^\Lambda d\xi \frac{\xi^{d-1}}{\beta c(2\pi)^d (d-2)}$$

Making the change of variables $y = q\xi$, and multiplying numerator and denominator by $\xi^2$, we find:

$$G(x, x) = \frac{S_d \xi^{2-d}}{\beta c(2\pi)^d} \int_0^{\Lambda \xi} dy \frac{y^{d-1}}{(y^2 + 1)} \approx \frac{S_d \xi^{2-d}}{\beta c(2\pi)^d (d-2)} = \frac{S_d \Lambda^{d-2}}{\beta c(2\pi)^d (d-2)}$$