Imagine a continuous curve in $d$-dimensional space given by the function $R(n)$, where the parameter $n$ runs from 0 to $N$ and measures the length along the curve in arbitrary units. We can associate a “stretching energy” to this curve in the form of a Hamiltonian functional:

$$H[R] = A \int_0^N \frac{dR(n)}{dn}^2$$

where $A > 0$ is a constant. Thus if you stretch (or compress) a certain part of the curve, making $R$ change more rapidly with $n$, you will increase the energy. This $H$ can be seen as a continuum description of a very simple polymer of fixed length $N$, and the partition function is given by the functional integral over all possible curves $R(n)$:

$$Z = \int DR e^{-\beta H[R]}$$

(a) We will “renormalize” this system by making a scale change: define a new parameter $n' = n/b$, which effectively means we are measuring lengths along the curves in larger units. Introduce a new curve function $R'(n') \equiv zR(n)$, where $z$ is some factor. Find the value of $z$ such that the Hamiltonian $H$ preserves its form under the transformation, becoming:

$$H'[R'] = A \int_0^{N'/b} \frac{dR'(n')}{dn'}^2$$

where $N' = N/b$.

**Answer:**

$$H = A \int_0^N \frac{dR}{dn}^2 = A \int_0^{N/b} \frac{1}{z} \frac{dR}{dn}^2 = A \int_0^{N/b} \frac{1}{z} \frac{dR'}{dn'}^2 = A \int_0^{N'/b} \frac{1}{z} \frac{dR'}{dn'}^2$$

Thus we need to have $z = b^{-1/2}$ for the Hamiltonian to preserve its form.

(b) The mean-squared end-to-end distance $\langle R^2 \rangle$ of the curve is given by:

$$\langle R^2 \rangle = \langle (R(N) - R(0))^2 \rangle = \frac{1}{Z} \int DR (R(N) - R(0))^2 e^{-\beta H[R]}$$

Using the result of part (a), show that the function $\langle R^2 \rangle$ obeys the relation: $\langle R^2 \rangle = b^\alpha \langle R^2 \rangle (N/b)$. Find the exponent $\alpha$. Use this relation to determine how $\langle R^2 \rangle$ scales with $N$.

**Answer:** Since the Hamiltonian preserves its form under the transformation, we can write:

$$\langle R^2 \rangle = \langle (R(N) - R(0))^2 \rangle = \langle (z^{-1} R'(N') - z^{-1} R'(0))^2 \rangle$$

$$= z^{-2} \langle (R'(N') - R'(0))^2 \rangle = z^{-2} \langle R^2 \rangle (N')$$

Plugging in $N' = N/b$ and $z = b^{-1/2}$, we get: $\langle R^2 \rangle = b \langle R^2 \rangle (N/b)$. Thus $\alpha = 1$. Letting $b = N$, we find: $\langle R^2 \rangle = N \langle R^2 \rangle (1) \propto N$. 