RG Methods in Statistical Field Theory: Quiz 5 Solution

Friday, November 3, 2006

Imagine a continuous curve in d-dimensional space given by the function $\mathbf{R}(n)$, where the parameter n runs from 0 to N and measures the length along the curve in arbitrary units. We can associate a "stretching energy" to this curve in the form of a Hamiltonian functional:

$$\mathcal{H}[\mathbf{R}] = A \int_0^N dn \left(\frac{d\mathbf{R}(n)}{dn} \right)^2$$

where $A > 0$ is a constant. Thus if you stretch (or compress) a certain part of the curve, making $\mathbf{R}$ change more rapidly with n, you will increase the energy. This $\mathcal{H}$ can be seen as a continuum description of a very simple polymer of fixed length N, and the partition function is given by the functional integral over all possible curves $\mathbf{R}(n)$:

$$Z = \int \mathcal{D}\mathbf{R}\, e^{-\beta \mathcal{H}[\mathbf{R}]}$$

(a) We will "renormalize" this system by making a scale change: define a new parameter $n' = n/b$, which effectively means we are measuring lengths along the curves in larger units. Introduce a new curve function $\mathbf{R}'(n') \equiv z\mathbf{R}(n)$, where z is some factor. Find the value of z such that the Hamiltonian $\mathcal{H}$ preserves its form under the transformation, becoming:

$$\mathcal{H}'[\mathbf{R}'] = A \int_0^{N'} dn' \left(\frac{d\mathbf{R}'(n')}{dn'} \right)^2$$

where $N' = N/b$.

Answer:

$$\begin{aligned}
\mathcal{H} = A \int_0^N dn \left(\frac{d\mathbf{R}}{dn} \right)^2 &= A \int_0^{N/b} dn'\, b \left(\frac{1}{z} \frac{d\mathbf{R}'}{dn} \right)^2 = A \int_0^{N/b} dn'\, b \left(\frac{1}{z} \frac{dn'}{dn} \frac{d\mathbf{R}'}{dn'} \right)^2 \\
&= A \int_0^{N/b} dn'\, b \left(\frac{1}{zb} \frac{d\mathbf{R}'}{dn'} \right)^2 = A \int_0^{N'} dn'\, \frac{1}{bz^2} \left(\frac{d\mathbf{R}'}{dn'} \right)^2
\end{aligned}$$

Thus we need to have $z = b^{-1/2}$ for the Hamiltonian to preserve its form.

(b) The mean-squared end-to-end distance $\bar{R}^2$ of the curve is given by:

$$\bar{R}^2 = \langle (\mathbf{R}(N) - \mathbf{R}(0))^2 \rangle = \frac{1}{Z} \int \mathcal{D}\mathbf{R}\, (\mathbf{R}(N) - \mathbf{R}(0))^2 e^{-\beta \mathcal{H}[\mathbf{R}]}$$

Using the result of part (a), show that the function $\bar{R}^2(N)$ obeys the relation: $\bar{R}^2(N) = b^\alpha \bar{R}^2(N/b)$. Find the exponent α. Use this relation to determine how $\bar{R}^2(N)$ scales with N.

Answer: Since the Hamiltonian preserves its form under the transformation, we can write:

$$\begin{aligned}
\bar{R}^2(N) = \langle (\mathbf{R}(N) - \mathbf{R}(0))^2 \rangle &= \langle (z^{-1}\mathbf{R}'(N') - z^{-1}\mathbf{R}'(0))^2 \rangle \\
&= z^{-2} \langle (\mathbf{R}'(N') - \mathbf{R}'(0))^2 \rangle = z^{-2} \bar{R}^2(N')
\end{aligned}$$

Plugging in $N' = N/b$ and $z = b^{-1/2}$, we get: $\bar{R}^2(N) = b\bar{R}^2(N/b)$. Thus $\alpha = 1$. Letting $b = N$, we find: $\bar{R}^2(N) = N\bar{R}^2(1) \propto N$.