Consider a quantum system of two sites, with one fermionic state at each site. The creation/destruction operators for this state are \( c_1^\dagger, c_1, c_2^\dagger, c_2 \) for sites 1 and 2 respectively. We have a Hamiltonian:

\[
H = -t(c_1^\dagger c_2 + c_2^\dagger c_1) - \mu(c_1^\dagger c_1 + c_2^\dagger c_2)
\]

The first term describes hopping between the sites, while the second term is the chemical potential. In the Fock space there are four basis states, which we label \(|n_1, n_2\rangle\), where \(n_i\) is the occupation number of site \(i\):

\[
|0, 0\rangle, \quad |1, 0\rangle = c_1^\dagger |0, 0\rangle, \quad |0, 1\rangle = c_2^\dagger |0, 0\rangle, \quad |1, 1\rangle = c_1^\dagger c_2^\dagger |0, 0\rangle,
\]

Find the partition function \(Z\) by any method you choose. **Hint:** The easiest way is to construct the Hamiltonian matrix in the Fock space basis, and then find its eigenvalues.

**Answer:** The operator \(H\) acting on the basis states gives the following results:

\[
H|0, 0\rangle = 0, \quad H|1, 0\rangle = -t|0, 1\rangle - \mu|1, 0\rangle, \quad H|0, 1\rangle = -t|1, 0\rangle - \mu|0, 1\rangle, \quad H|1, 1\rangle = -2\mu|1, 1\rangle
\]

Thus \(H\) corresponds to the following matrix (with the rows and columns in the same order as the basis states are listed above):

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -\mu & -t & 0 \\
0 & -t & -\mu & 0 \\
0 & 0 & 0 & -2\mu
\end{pmatrix}
\]

The eigenvalues of this matrix (the eigenenergies \(E_n\) of the system) are: 0, \(-\mu - t\), \(-\mu + t\), \(-2\mu\). Thus the partition function is given by:

\[
Z = \sum_n e^{-\beta E_n} = 1 + e^{\beta(\mu + t)} + e^{\beta(-\mu - t)} + e^{2\beta\mu}
\]