Consider a lattice spinless fermion system described by the following Hamiltonian:

\[ \mathcal{H} = - t \sum_{\langle ij \rangle} (c_i^\dagger c_j + c_j^\dagger c_i) - \mu \sum_i c_i^\dagger c_i \]

It contains only nearest-neighbor hopping and chemical potential terms. The partition function can be written as a Grassmann path integral:

\[ Z = \int e^{\beta S} \prod_{n=-\infty}^{\infty} \prod_i d\bar{\psi}_i(\omega_n) d\psi_i(\omega_n) \]

where \( \omega_n \) are the Matsubara frequencies. Write an expression for the action \( S \). (You can just give the answer without any derivation.)

**Answer:**

\[ S = \beta \sum_n \left[ \sum_i i \omega_n \bar{\psi}_i(\omega_n) \psi_i(\omega_n) + t \sum_{\langle ij \rangle} (\bar{\psi}_i(\omega_n) \psi_j(\omega_n) + \bar{\psi}_j(\omega_n) \psi_i(\omega_n)) + \mu \sum_i \bar{\psi}_i(\omega_n) \psi_i(\omega_n) \right] \]

The first term is the Fourier transformed version of \( -\bar{\psi}(\tau) \cdot \frac{d}{d\tau} \psi(\tau) \), while the second and third terms are just \( -\mathcal{H} \) with the creation and destruction operators replaced by \( \bar{\psi} \) and \( \psi \) Grassmann vectors.